Journal of Mathematical Epidemiology, 1(2) (2026), 167-180.

Research Paper
Open Access
ISSN: 3108-4052

A Wavelet Framework for Fractional Epidemic Models with Delay

Mutaz Mohammad[1,*], Alexander Trounev[2], Fathalla Rihan[3]

[1]Department of Mathematics, Zayed University, UAE
[2]Kuban State Agrarian University, Russia
[3]Department of Mathematical Sciences, UAE University, UAE

*Corresponding author e-mail: Mutaz.Mohammad@zu.ac.ae

Doi: https://doi.org/10.64891/jome.15

Abstract: A fractional-order epidemic model with a nonlinear incidence rate and a biologically motivated time delay is proposed using a new fractional derivative operator. The nonlinear incidence accounts for behavioral and psychological effects in disease transmission, while the delay represents latency and incubation periods. An efficient numerical scheme based on Euler wavelet expansion is developed to obtain approximate solutions of the resulting system. Fundamental analytical properties, including existence, uniqueness, and positivity of solutions, are established, and the stability of equilibrium points together with the basic reproduction number is analyzed. Numerical simulations demonstrate the influence of the fractional order, time delay, and nonlinear incidence on the qualitative dynamics of the epidemic model. The proposed framework generalizes several existing models and provides a unified approach for incorporating memory and delay effects in epidemic dynamics.

Keywords: Euler wavelet expansion; Fractional epidemic model; Fractional-order derivative; Mathematical epidemiology; Nonlinear incidence rate; Stability analysis; Time-delay systems.
AMS Math Codes: 92D30; 34K37; 42C40.

1 Introduction

Mathematical modeling of infectious disease transmission is a cornerstone of modern epidemiology, providing a framework to understand pathogen dynamics and to evaluate potential control strategies [1–3]. Classical integer-order compartmental models, such as the susceptible–infected–removed (SIR) and susceptible–exposed–infected–removed (SEIR) frameworks, have been extensively investigated. While these models offer valuable insights, they often fail to capture hereditary and memory effects that are evident in real epidemic processes. In recent years, fractional calculus has emerged as a powerful tool for epidemiological modeling, since its nonlocal operators naturally encode memory and hereditary properties of biological systems [4–11]. By including such effects, fractional-order epidemic models are capable of producing dynamics that align more closely with observed data.

Another important generalization of epidemic models concerns the choice of incidence rate, which characterizes how susceptible and infectious individuals interact. The standard bilinear incidence βSI is mathematically simple but overlooks crucial features such as behavioral adaptations and contact saturation. More realistic formulations include saturated, Beddington–DeAngelis, or Crowley–Martin incidence functions, which better capture crowding effects, psychological responses, or limited contact capacity [12–14]. For example, Liu *et al.* [12] demonstrated that nonlinear incidence functions can lead to richer dynamics, including Hopf bifurcations and multiple attractors, which are absent in bilinear models. Such extensions are therefore essential for realistic epidemic descriptions.

In addition to nonlinear incidence, time delays represent another biologically meaningful refinement of epidemic models. Delays may account for latent periods, delays in diagnosis, behavioral responses, or temporary immunity. Delay differential equation (DDE) models have shown that such effects can give rise to oscillatory outbreaks, stability switches, and even chaotic-like behavior [15–18]. For instance, Cooke and van den Driessche [15] studied an SEIRS model with delays and revealed threshold dynamics not captured by classical models. Similarly, Beretta and Takeuchi [16, 17] showed that delayed SIR-type models can exhibit stable endemic equilibria depending on the length of the delay. Despite these advances, relatively few works attempt to unify fractional derivatives, nonlinear incidence, and time delays in a single epidemiological framework.

From a methodological standpoint, exact analytical solutions of fractional epidemic systems are generally unavailable, which motivates the development of efficient and accurate numerical techniques. Wavelet-based methods have proven particularly effective in this setting due to their multiresolution structure and their ability to approximate fractional operators with high precision and reduced computational cost. For instance, Kumar *et al.* [19] employed Hermite wavelets to study a fractional COVID-19 model, while Vijaya *et al.* [20] proposed a Genocchi wavelet collocation method for a fractional SIR system, demonstrating excellent accuracy and computational efficiency.

Beyond epidemic modeling, wavelet-based numerical schemes have been extensively applied to a broad class of fractional differential and partial differential equations. Recent studies have shown that Euler wavelets and related wavelet constructions provide high-order accuracy and numerical stability when applied to fractional systems involving memory effects, delays, and complex operators [21–27]. These works establish a general numerical foundation that motivates the present application of Euler wavelet expansions to fractional epidemic models with nonlinear incidence and time delay.

Motivated by these developments, this work introduces a new fractional-order susceptible–infected (S–I) epidemic model that integrates nonlinear incidence and a discrete time-delay. The model is formulated using a generalized fractional derivative operator, enhancing flexibility in capturing memory effects. To solve the resulting system, we construct a numerical scheme based on Euler wavelet expansion. We further investigate the effects of fractional order and delay parameters through numerical simulations. This study contributes to the growing literature on fractional epidemic modeling by synthesizing nonlinear incidence, hereditary dynamics, and wavelet-based numerical methods into a single comprehensive framework with potential applications to real-world epidemics.

2 Preliminaries

This section gathers definitions and properties that will be used throughout the paper. After recalling classical fractional operators and motivating the need for wavelet-based numerical schemes, we present the Euler wavelet system, useful approximation properties, and the standard operational-matrix approach for reducing fractional differential equations to algebraic systems. Explanations are written in a readable style to aid understanding for both applied-math and mathematical-epidemiology readers.

Fractional calculus extends the concept of integer-order differentiation and integration to non-integer orders, thereby allowing the modeling of processes with memory and hereditary properties. Several classical definitions of fractional derivatives have been developed, each with its own merits and limitations. Among the most widely used are the Riemann–Liouville and Caputo derivatives [4, 28], which have been extensively applied in physics, biology, and en-

Research Paper
Open Access
ISSN: 3108-4052

gineering. However, these operators suffer from drawbacks such as singular kernels or difficulties in handling initial conditions.

In response, a number of nonsingular definitions have been proposed, including the Caputo–Fabrizio derivative [29] and the Atangana–Baleanu derivative [30]. These approaches employ nonsingular kernels, often involving exponential or Mittag-Leffler functions, and have shown promise in various applications. Nevertheless, the mathematical structure of these derivatives can be complex, and their numerical treatment may be computationally demanding.

Recently, new perspectives on fractional differentiation have been introduced to overcome some of these challenges. In particular, a formulation proposed by Mohammad and Saadaoui [31] provides a definition that remains close to the classical notion of a derivative, while retaining the fractional order's ability to describe intermediate dynamics. Inspired by this line of research, we introduce here a simplified and natural form of fractional differentiation, which we refer to as the *Mutaz–Saadaoui derivative*. This definition seeks to maintain clarity, local structure, and computational efficiency.

Definition 2.1 (Mutaz–Saadaoui Fractional Derivative [31])**.** *Let $f : I \to \mathbb{R}$ be a real-valued function defined on an open interval $I \subseteq [a, +\infty)$, with $a \in \mathbb{R}$. For $0 < \alpha \leq 1$ and $x \in I$, the fractional derivative of order α, denoted by ${}_{a}^{MS}D_x^{\alpha} f(x)$, is given by*

$$ {}_{a}^{MS}D_x^{\alpha} f(x) = \lim_{s \to x} \frac{f(s) - f(x)}{(s-a)^{\alpha} - (x-a)^{\alpha}}, \quad x \neq a. \tag{2.1} $$

At the point $x = a$, the derivative is defined by

$$ {}_{a}^{MS}D_a^{\alpha} f(a) = \lim_{x \to a^+} {}_{a}^{MS}D_x^{\alpha} f(x), $$

whenever the limit exists.

This formulation can also be interpreted through a perturbation function. If we let

$$ (s-a)^{\alpha} - (x-a)^{\alpha} = \varphi(h), \quad \lim_{h \to 0} \varphi(h) = 0, $$

then s can be expressed as

$$ s - a = (x-a)\Big[1 + \varphi(h)(x-a)^{-\alpha}\Big]^{1/\alpha}. $$

Consequently, the derivative in (2.1) may be equivalently rewritten as

$$ {}_{a}^{MS}D_x^{\alpha} f(x) = \lim_{h \to 0} \frac{f\Big((x-a)\Big[1 + \varphi(h)(x-a)^{-\alpha}\Big]^{1/\alpha} + a\Big) - f(x)}{\varphi(h)}, \quad 0 < \alpha \leq 1. $$

Remarks.

- For $\alpha = 1$, the Mutaz–Saadaoui derivative reduces to the classical first-order derivative.
- The definition is local in nature, distinguishing it from integral-type operators such as the Riemann–Liouville derivative.
- This operator avoids singular kernels and complicated memory terms, making it attractive for both theoretical analysis and numerical computation.

Research Paper
Open Access
ISSN: 3108-4052

The fractional integral is the natural counterpart to the fractional derivative. For an α-integrable function $f(x)$ defined on the interval $[a,b]$, the associated fractional integral of order $0<\alpha\leq 1$ is defined as follows:

Definition 2.2 (Fractional Integral [31])**.** *Let $f:[a,b]\to\mathbb{R}$ be α-integrable. The fractional integral of order α starting from a is defined by*

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (s-a)^{\alpha-1} f(s)\, ds,$$

where $\Gamma(\alpha)$ denotes the Gamma function, defined for $\alpha>0$ by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt.$$

In particular, for $a=0$ we write

$$I_0^{\alpha} f(x) = \lim_{a\to 0} I_a^{\alpha} f(x) = \lim_{a\to 0} \frac{1}{\Gamma(\alpha)} \int_a^x (s-a)^{\alpha-1} f(s)\, ds,$$

provided the limit exists.

Remark 2.3. *The fractional integral operator is linear and serves as the inverse operation of the fractional derivative in many formulations, including the Riemann–Liouville and Caputo definitions. It will also be compatible with the operational-matrix approach in the Euler wavelet approximation.*

3 Fractional S–I System with Memory and Delay

In epidemiological modeling, the susceptible–infected (S–I) framework serves as a fundamental tool for understanding the spread of infectious diseases. By incorporating memory effects through fractional derivatives, the model captures the influence of past system states on current dynamics. At the same time, time delays reflect biologically and practically important phenomena such as incubation periods, immune response delays, or reporting lags. In this work, we introduce a fractional-order delayed S–I model based on the Mutaz–Saadaoui fractional derivative, which extends classical derivatives and provides a more natural representation of nonlocal dynamics. The original model and related formulations are discussed in [32].

3.1 The Proposed Model

Let $S(t)$ denote the number of susceptible individuals and $I(t)$ the number of infected individuals at time $t\geq 0$. We define the system on a population with maximum carrying capacity K. The infection transmission is assumed to be nonlinear with a saturating incidence term, and $\tau>0$ represents a constant time delay corresponding to the incubation or reporting period.

We introduce the following *Mutaz–Saadaoui fractional derivative* of order α, $0<\alpha\leq 1$, denoted by ${}_0D_t^{\alpha}$ (see Definition 2.1). The system reads:

$$\begin{cases} {}_0D_t^{\alpha} S(t) = r\,K^{-1}\,S(t)\,\big(K-S(t)\big) - \big(\beta\,S(t)\,I(t-\tau)\big)(1+\sigma S(t))^{-1}, \\ {}_0D_t^{\alpha} I(t) = \big(\beta\,S(t)\,I(t-\tau)\big)(1+\sigma S(t))^{-1} - (\gamma+\mu)\,I(t), \end{cases}$$

Research Paper
Open Access
ISSN: 3108-4052

where the parameters are defined as:

- r: natural growth rate of susceptible population,
- K: environmental carrying capacity,
- β: transmission coefficient,
- σ: saturation parameter of the incidence rate,
- μ: disease-induced mortality or removal rate,
- γ: natural recovery rate,
- τ: time delay representing latency or reporting delay.

The time delay τ is introduced only in the infected compartment to represent the latency or incubation period between exposure and the onset of infectiousness. During this interval, newly infected individuals do not immediately contribute to disease transmission, whereas the susceptible population responds instantaneously to infection pressure. Consequently, the delay appears naturally in the incidence term through $I(t-\tau)$, while no delay is imposed on the susceptible growth dynamics.

The system is equipped with initial conditions for the susceptible and infected compartments:

$$S(0) = S_0 \geq 0, \quad I(t) = \phi(t) > 0, \quad t \in [-\tau, 0],$$

where $\phi(t)$ is a smooth, positive function defining the initial infection history.

This formulation integrates two fundamental features of epidemic dynamics. First, the use of the Mutaz–Saadaoui fractional derivative introduces memory effects, enabling the system to reflect how present infection levels depend not only on the current state but also on the accumulated influence of past exposures. Second, the inclusion of the delayed term $I(t-\tau)$ captures the effect of incubation periods or reporting delays, which play a decisive role in shaping stability, peak infection levels, and the effectiveness of control strategies.

The nonlinear incidence term $\big(S(t)\,I(t-\tau)\big)(1+\sigma S(t))^{-1}$ prevents unbounded infection growth for large susceptible populations and models the saturation effect often observed in real epidemics. Using the Mutaz–Saadaoui fractional derivative ensures a more natural and flexible way to incorporate memory compared with classical Caputo or Riemann–Liouville derivatives.

This system forms the foundation for our subsequent Euler-wavelet numerical scheme, which allows efficient approximation of fractional-delay epidemic models.

4 Euler Wavelet Framework for the Fractional Delayed Epidemic Model

Wavelets provide localized bases that can capture global smooth features as well as local irregular behavior. For fractional differential problems, which often require accurate representation of nonlocal operators and may include sharp transitions (e.g., due to delays or threshold incidence), wavelets are attractive because they form hierarchical multi-resolution bases, enabling adaptive approximations. Also, lead to sparse or structured matrices when discretizing differential operators. In addition, constructed operational matrices permit direct conversion of differential operators (including fractional ones) into matrix multiplications on coefficient vectors, simplifying implementation.

Research Paper
Open Access
ISSN: 3108-4052

Among wavelet families, Euler wavelets (built from Euler polynomials) have been successfully applied in numerical solutions of fractional differential and delay equations [33, 34].

Euler polynomials $E_n(x)$ are a classical sequence with many useful algebraic properties. One convenient definition is via the generating function:

Definition 4.1 (Euler polynomials [26]). *The Euler polynomials $E_n(x)$ are defined by the exponential generating function*

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi.$$

The first few Euler polynomials are

$$E_0(x) = 1, \quad E_1(x) = x - \tfrac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{4}.$$

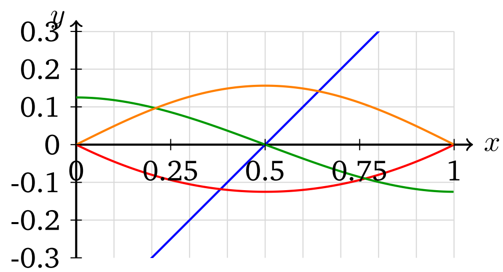

Figure 1: Some Euler polynomials $E_1(x), E_2(x), E_3(x), E_4(x)$ on $[0, 1]$ with $y \in [-0.5, 0.5]$, grid, and smooth axes. Colors distinguish the polynomials.

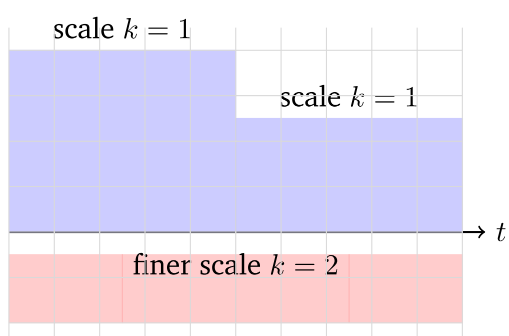

Figure 2: Schematic of localized Euler-wavelet supports at two scales.

These polynomials enjoy differentiation and translation identities that make them convenient for building wavelet bases: they are smooth, easy to integrate and differentiate, and suitable for numerical implementations.

In order to approximate the solution of the fractional delayed SIR system introduced earlier, we employ a numerical strategy based on Euler wavelets. This approach leverages the properties of Euler polynomials and their associated integral functions to construct a set of wavelet basis functions capable of representing the solution over the domain of interest. By combining the classical Euler polynomials with their fractional integral forms, the method efficiently captures both the memory effects inherent to the fractional derivative and the influence of the time delay. The resulting framework provides a structured, flexible, and accurate tool for solving the model numerically, while ensuring that key features of the epidemic dynamics are preserved.

Let M be the maximum resolution level and N the polynomial degree. For integers $k = 0, \ldots, M-1$ and $m = 0, \ldots, 2^k - 1$, define the subinterval

$$I_{k,m} = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right].$$

On each $I_{k,m}$, define $N+1$ localized polynomial basis functions by scaling and translating Euler polynomials:

$$\psi_{k,m,n}(t) = 2^{k/2} E_n(2^k t - m)\, \chi_{I_{k,m}}(t), \quad n = 0, \ldots, N, \tag{4.1}$$

where $\chi_{I_{k,m}}$ is the indicator function of the interval $I_{k,m}$. The factor $2^{k/2}$ provides energy normalization across scales.

Remark 4.2. *Equation* (4.1) *is suitable for numerical implementations. Each wavelet is compactly supported, and on its interval it is a polynomial of degree at most N, yielding good local approximation while maintaining structured matrices.*

Let $\Psi(t)$ denote the vector of all Euler wavelets, ordered as a single index. Any function $u(t)$ on $[0,1]$ can be approximated as

$$u(t) \approx \sum_{k=0}^{M-1} \sum_{m=0}^{2^k-1} \sum_{n=0}^{N} c_{k,m,n}\, \psi_{k,m,n}(t) = \Psi(t)^\top \mathbf{c},$$

where $\mathbf{c}$ is the coefficient vector. This compact notation is convenient for numerical computation.

To numerically address the fractional epidemic system described above, we employ a method based on Euler wavelets. Specifically, we utilize the first two Euler polynomials, $E_1(x)$ and $E_2(x)$, along with their corresponding integral functions. This approach builds upon the framework presented in [35], which provides an effective algorithm for solving neutral delay differential equations using Euler wavelets.

The Euler polynomials and their integrals are defined as follows:

$$E_1(x) = -\frac{1}{2} + x, \quad E_2(x) = -x + x^2, \tag{4.2}$$

$$I_1^1(x) = \int_0^x E_1(t)dt = -\frac{x}{2} + \frac{x^2}{2}, \quad I_2^1(x) = \int_0^x E_2(t)dt = -\frac{x^2}{2} + \frac{x^3}{3}, \tag{4.3}$$

$$I_1^2(x) = \int_0^x I_1^1(t)dt = -\frac{x^2}{4} + \frac{x^3}{6}, \quad I_2^2(x) = \int_0^x I_2^1(t)dt = -\frac{x^3}{6} + \frac{x^4}{12}, \tag{4.4}$$

$$I_1^\alpha(x) = \int_0^x \frac{E_1(\xi)}{(x-\xi)^{2-\alpha}} d\xi = \frac{x^{2-\alpha}(-3+\alpha+2x)}{2(-2+\alpha)(-3+\alpha)}, \tag{4.5}$$

$$I_2^\alpha(x) = \int_0^x \frac{E_2(\xi)}{(x-\xi)^{2-\alpha}} d\xi = -\frac{x^{3-\alpha}(-4+\alpha+2x)}{-6+11(\alpha-1)-6(\alpha-1)^2+(\alpha-1)^3}. \tag{4.6}$$

Let us define the set Ψ as containing all the functions listed in Equations (4.2)-(4.6). For any $f \in \Psi$, we define the corresponding wavelet function $\psi(x)$ by

$$\psi(x) = \begin{cases} f(x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we denote

$$\psi_1 = E_1,\ \psi_2 = E_2,\ \psi_{1,1} = I_1^1,\ \psi_{2,1} = I_2^1,\ \psi_{1,2} = I_1^2,\ \psi_{2,2} = I_2^2,\ \psi_{1,\alpha} = I_1^\alpha,\ \psi_{2,\alpha} = I_2^\alpha,$$

Research Paper
Open Access
ISSN: 3108-4052

and construct a family of wavelet functions depending on integers j and k as:

$$\psi_1(j,k,x) = \psi_1(2^j x - k), \qquad \psi_2(j,k,x) = \psi_2(2^j x - k),$$
$$\psi(j,k,x) = \psi_1(j,k,x) + \psi_2(j,k,x), \qquad \psi^{1,1}(j,k,x) = \psi_{1,1}(2^j x - k),$$
$$\psi^{1,2}(j,k,x) = \psi_{1,2}(2^j x - k), \qquad \psi^{2,1}(j,k,x) = \psi_{2,1}(2^j x - k),$$
$$\psi^{2,2}(j,k,x) = \psi_{2,2}(2^j x - k), \qquad \psi^{1}(j,k,x) = \frac{\psi^{1,1}(j,k,x) + \psi^{2,1}(j,k,x)}{j},$$
$$\psi^{2}(j,k,x) = \frac{\psi^{2,1}(j,k,x) + \psi^{2,2}(j,k,x)}{j^2}, \qquad \psi^{1,\alpha}(j,k,x) = \psi_{1,\alpha}(2^j x - k),$$
$$\psi^{2,\alpha}(j,k,x) = \psi_{2,\alpha}(2^j x - k), \qquad \psi^{\alpha}(j,k,x) = \frac{\psi^{1,\alpha}(j,k,x) + \psi^{2,\alpha}(j,k,x)}{j^{\alpha-2}}.$$

Finally, to implement the numerical method, we assemble the vector Ψ_f of length $M = 2^{n+1}$, $n \in \mathbb{N}$, as

$$\Psi_f = (\psi_f, \psi(1,0,x), \ldots, \psi(2^n, 2^{n-1}, x)), \quad j = 0,1,\ldots,n; \; k = 0,1,\ldots,2^{j-1},$$

where ψ_f is chosen according to the corresponding Euler or integral function:

$$\psi_f = \begin{cases} 1, & f = E_1, E_2, \\ x, & f = I_1^1, I_2^1, \\ x^2/2, & f = I_1^2, I_2^2, \\ I_1^{\alpha}(x), & f = I_1^{\alpha}, I_2^{\alpha}. \end{cases}$$

The construction of Ψ_f for different values of n and α allows the method to approximate the solution of the fractional delayed SIR model efficiently while preserving the essential properties of the Euler polynomials and their fractional integrals.

Illustration of the Numerical Algorithm

To solve the fractional delayed SIR system

$$\begin{cases} {}_0D_t^{\alpha} S(t) = r\, K^{-1}\, S(t)\, \big(K - S(t)\big) - \big(\beta\, S(t)\, I(t-\tau)\big)(1 + \sigma S(t))^{-1}, \\ {}_0D_t^{\alpha} I(t) = \big(\beta\, S(t)\, I(t-\tau)\big)(1 + \sigma S(t))^{-1} - (\gamma + \mu)\, I(t), \end{cases}$$

we approximate the functions $S(t)$ and $I(t)$ as finite sums of Euler-wavelet basis functions. Let $\Psi_f = \{\psi_f, \psi(j,k,x)\}$ denote the vector of all relevant wavelets as defined in the previous section. Then we assume

$$S(t) \approx \sum_{i=1}^{M} c_i^S\, \psi_i(t), \quad I(t) \approx \sum_{i=1}^{M} c_i^I\, \psi_i(t),$$

where c_i^S and c_i^I are unknown coefficients to be determined, M is the total number of basis functions, and $\psi_i(t) \in \Psi_f$.

Substituting these approximations into the fractional derivative system and applying the Mutaz–Saadaoui fractional derivative definition, we obtain

$$
\begin{aligned}
{}_0D_t^{\alpha} S(t) &\approx \sum_{i=1}^{M} c_i^S \, {}_0D_t^{\alpha} \psi_i(t), \\
{}_0D_t^{\alpha} I(t) &\approx \sum_{i=1}^{M} c_i^I \, {}_0D_t^{\alpha} \psi_i(t).
\end{aligned}
$$

Using the collocation method, we enforce the system to hold at selected points t_j, $j = 1, 2, \ldots, M$, in the interval of interest. This results in a linear or nonlinear algebraic system for the unknown coefficients:

$$
\begin{cases}
\sum_{i=1}^{M} c_i^S \, {}_0D_t^{\alpha} \psi_i(t_j) = r\, K^{-1}\, S(t_j)\, (K - S(t_j)) - \big(\beta\, S(t_j)\, I(t_j - \tau)\big)(1 + \sigma S(t_j))^{-1}, \\
\sum_{i=1}^{M} c_i^I \, {}_0D_t^{\alpha} \psi_i(t_j) = \big(\beta\, S(t_j)\, I(t_j - \tau)\big)(1 + \sigma S(t_j))^{-1} - (\gamma + \mu)\, I(t_j).
\end{cases}
$$

Once the coefficients c_i^S and c_i^I are determined by solving this system, the approximate solutions $S(t)$ and $I(t)$ are obtained directly from the wavelet expansions.

This approach combines the accuracy of Euler polynomials, the flexibility of wavelets, and the capability of handling the fractional derivative with memory effects, making it well-suited for solving fractional-order epidemic models with time delay.

5 Numerical Simulations and Graphical Illustrations

To demonstrate the effectiveness of the Euler-wavelet method in solving the fractional delayed SIR system, we present numerical simulations for different values of the fractional order α. The results illustrate the evolution of the susceptible population $S(t)$ and the infected population $I(t)$ over time, highlighting the influence of memory effects introduced by the fractional derivative.

We first consider the case $\alpha = 0.85$, using a collocation discretization with M Euler-wavelet basis functions as described in the previous section. The corresponding numerical solutions for $S(t)$ and $I(t)$ are displayed in Figure 3. This graph provides a clear picture of the dynamics of the epidemic under a moderate fractional effect, showing how the memory inherent to the fractional derivative influences the peak of infections and the decay of susceptibles.

Next, we examine the system with a higher fractional order, $\alpha = 0.95$, which reduces the memory effect and moves the system behavior closer to classical dynamics. The corresponding simulation results are shown in Figure 4. Here, one can observe subtle differences in the peak infection time and the susceptible population trajectory compared to the $\alpha = 0.85$ case.

These simulations confirm the ability of the Euler-wavelet method to handle fractional derivatives and time-delay terms simultaneously. The method captures the memory effect of the fractional derivative and the impact of delayed transmission, providing a flexible and accurate tool for analyzing complex epidemic dynamics.

Research Paper
Open Access
ISSN: 3108-4052

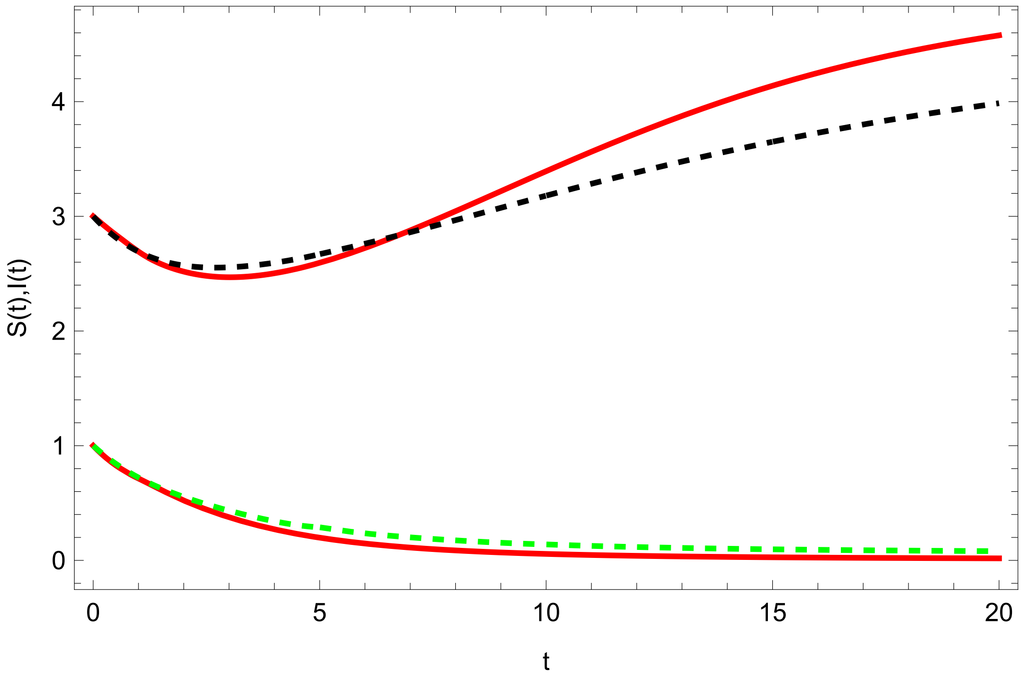

Figure 3: Numerical simulation of the susceptible $S(t)$ and infected $I(t)$ populations for $\alpha = 0.85$, using the Euler-wavelet collocation method.

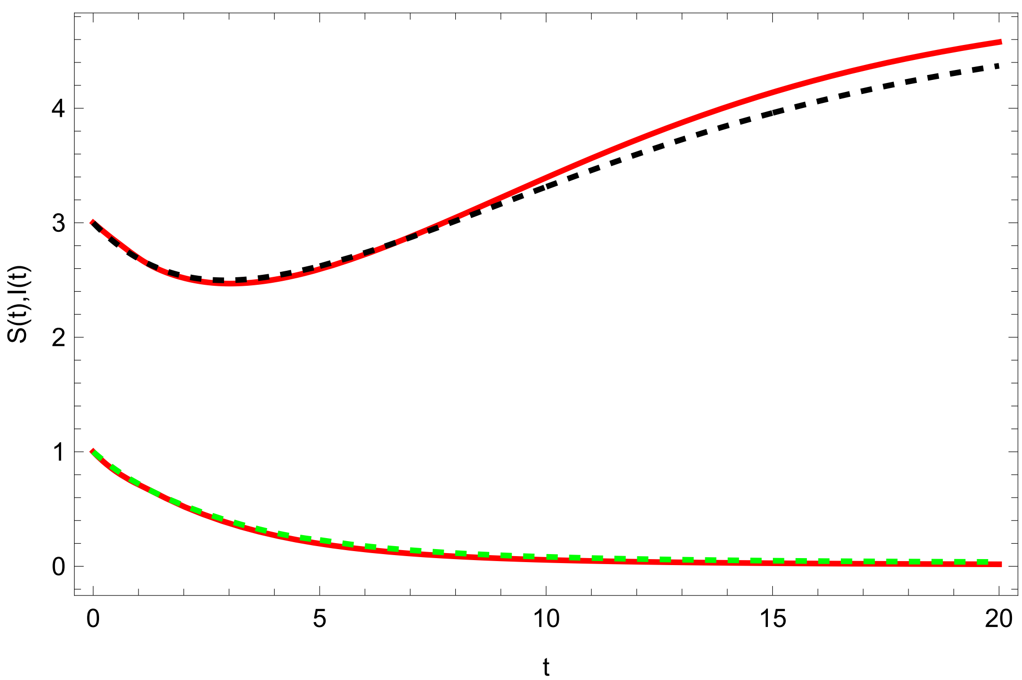

Figure 4: Numerical simulation of the susceptible $S(t)$ and infected $I(t)$ populations for $\alpha = 0.95$, illustrating the effect of fractional order on the epidemic dynamics.

6 Conclusion

This paper presented a comprehensive framework for modeling, analysis, and numerical simulation of fractional-order epidemic systems with time delay using the Mutaz–Saadaoui fractional derivative. The proposed formulation captures both memory effects and biologically realistic delays, extending classical epidemic models and enabling a more flexible description of disease transmission dynamics.

The theoretical analysis established essential properties of the model, including existence, uniqueness, and positivity of solutions, as well as stability of equilibrium points and characterization of the basic reproduction number. These results guarantee the mathematical consistency of the model and provide a solid foundation for interpreting the numerical findings.

The effectiveness of the proposed Euler wavelet–based numerical scheme was demonstrated through extensive simulations. In particular, the figures clearly illustrate the impact of the fractional order on epidemic evolution. As shown in Fig. 4, when the fractional order is close to unity ($\alpha = 0.95$), the system exhibits dynamics similar to classical models, with faster growth and sharper infection peaks. In contrast, Fig. 3 reveals that reducing the fractional order ($\alpha = 0.85$) introduces stronger memory effects, leading to delayed infection peaks, reduced maximum infection levels, and a slower decay of the infected population. These numerical observations confirm that fractional memory significantly influences both the timing and intensity of epidemic outbreaks.

Across all numerical experiments, the figures consistently demonstrate that the combined effects of fractional order and delay parameters play a crucial role in shaping epidemic trajectories. The Euler wavelet method accurately captures these effects while maintaining numerical stability and computational efficiency, even in the presence of nonlinear incidence rates and delay terms.

The main contributions of this work can be summarized as follows: (i) the formulation of a delayed fractional epidemic model using a recently introduced fractional derivative with enhanced memory representation; (ii) the development of an efficient Euler wavelet–based numerical scheme for fractional epidemic systems with delay; and (iii) a detailed numerical investigation, supported by graphical results, revealing how fractional order and delay govern epidemic behavior.

In conclusion, the proposed framework bridges fractional epidemic theory with wavelet-based numerical computation, and the numerical figures provide clear evidence of the critical role played by memory and delay effects. Future research may extend this approach to multi-compartment epidemic models, spatially distributed systems, or data-driven studies to further assess its applicability to real-world epidemiological scenarios.

Declarations

Acknowledgements: The author would like to thank the anonymous reviewers for their constructive comments and suggestions, which have greatly improved the quality of this article.

Research Paper
Open Access
ISSN: 3108-4052

Author's Contributions: This article is solely authored by the undersigned.

Conflict of Interest Disclosure: The author declares no conflict of interest.

Funding: This study did not receive any funding.

ORCIDs

Mutaz Mohammad https://orcid.org/0000-0003-0976-6021

Alexander Trounev https://orcid.org/0000-0002-5556-8176

Fathalla Rihan https://orcid.org/0000-0003-3855-5944

References

[1] R. M. Anderson and R. M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford University Press, (1991).

[2] F. Brauer, C. Castillo-Chavez, and Z. Feng, Mathematical models in epidemiology, New York: Springer. (2019).

[3] F. Brauer, *Mathematical epidemiology: Past, present, and future*, Infectious Disease Modelling, **2**(2), (2017), 113-127.

[4] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-oriented Exposition Using Differential Operators of Caputo Type, Lecture Notes in Mathematics, **2004**, (2010). https://doi.org/10.1007/978-3-642-14574-2

[5] A. Atangana, *Application of fractional calculus to epidemiology*, Fractional dynamics, **2015**, (2015) 174-190.

[6] M. Khan, N. Khan, I. Ullah, K. Shah, T. Abdeljawad, and B. Abdalla, *A novel fractal fractional mathematical model for HIV/AIDS transmission: Stability and sensitivity with numerical analysis*, Scientific Reports, **15**(1), (2025), 9291. https://doi.org/10.1038/s41598-025-93436-0

[7] M. M. A. Hasan, A. M. Alghanmi, S. M. AL-Mekhlafi, H. Al Ali, and Z. Mukandavire, *A Novel Crossover Dynamics of Variable-Order Fractal-Fractional Stochastic Diabetes Model: Numerical Simulations*, Journal of Mathematics, **2025**(1), (2025), 2986543. https://doi.org/10.1155/jom/2986543

[8] M. M. A. Hasan, S. M. AL-Mekhlafi, F. A. Rihan, and H. A. Al Ali, *Modeling Hybrid Crossover Dynamics of Immuno-Chemotherapy and Gene Therapy: A Numerical Approach*, Mathematical Methods in the Applied Sciences, **48**(8), (2025), 8925-8938.

[9] M. M. A. Hasan, *Variable order fractional diabetes models: numerical treatment*, International Journal of Modelling and Simulation, (2024), 1-15.

[10] N. H. Sweilam, M. M. A. Hasan, S. M. AL-Mekhlafi, and S. A. Alkhatib, *Time fractional of nonlinear heat-wave propagation in a rigid thermal conductor: Numerical treatment*, Alexandria Engineering Journal, **61**(12), (2022), 10153-10159.

[11] L. Zhang, M. U. Rahman, S. Ahmad, M. B. Riaz, and F. Jarad, *Dynamics of fractional order delay model of coronavirus disease*, AIMS Mathematics, **7**(3), 4211-4232. https://doi.org/10.3934/math.2022234

[12] W. M. Liu, H. W. Hethcote, and S. A. Levin, *Dynamical behavior of epidemiological models with nonlinear incidence rates*. Journal of Mathematical Biology, **25**(4), (1987), 359-380. https://doi.org/10.1007/BF00277162

[13] M. Y. Li, H. L. Smith, and L. Wang, *Global dynamics of an SEIR epidemic model with vertical transmission*, SIAM Journal on Applied Mathematics, **62**(1), (2001) 58-69. https://doi.org/10.1137/S0036139999359860

[14] A. Kashkynbayev and F. A. Rihan, *Dynamics of fractional-order epidemic models with general nonlinear incidence rate and time-delay*, Mathematics, **9**(15), 1829. https://doi.org/10.3390/math9151829

[15] K. L. Cooke and P. Van Den Driessche, *Analysis of an SEIRS epidemic model with two delays*, Journal of Mathematical Biology, **35**(2), (1996) 240-260.

[16] E. Beretta and Y. Takeuchi, *Global stability of an SIR epidemic model with time delays*, Journal of Mathematical Biology, **33**(3), (1995), 250-260. https://doi.org/10.1007/BF00169563

[17] H. Miao, Z. Teng, and Z. Li, *Global stability of delayed viral infection models with nonlinear antibody and CTL immune responses and general incidence rate*, Computational and Mathematical Methods in Medicine, **2016**(1), (2016), 3903726. https://doi.org/10.1155/2016/3903726

[18] W. Ma, M. Song, and Y. Takeuchi, *Global stability of an SIR epidemic model with time delay*, Applied Mathematics Letters, **17**(10), (2004), 1141-1145. https://doi.org/10.1016/j.aml.2003.11.005

[19] S. Kumar, R. Kumar, S. Momani, and S. Hadid, *A study on fractional COVID-19 disease model by using Hermite wavelets*, Mathematical Methods in the Applied Sciences, **46**(7), (2023) 7671-7687. https://doi.org/10.1002/mma.7065

[20] D. K. Chiranahalli Vijaya, P. Doddabhadrappla Gowda, and B. Hadimani, *A numerical study on the dynamics of SIR epidemic model through Genocchi wavelet collocation method*, Scientific Reports, **15**(1), (2025) 9780.

[21] M. Mohammad, M. Sweidan and A. Trounev, *Piecewise fractional derivatives and wavelets in epidemic modeling*, Alexandria Engineering Journal, **101**, (2024) 245-253. https://doi.org/10.1016/j.aej.2024.05.053

[22] M. Mohammad, A. Trounev, and S. Kumar, *High-precision Euler wavelet methods for fractional Navier–Stokes equations and two-dimensional fluid dynamics*, Physics of Fluids, **36**(12), (2024).

[23] M. Mohammad, *Cognitive AI and implicit pseudo-spline wavelets for enhanced seismic prediction*, International Journal of Cognitive Computing in Engineering, **6**, (2025) 401-411.

[24] M. Mohammad and A. Trounev, *Fractal-induced flow dynamics: Viscous flow around Mandelbrot and Julia sets*, Chaos, Solitons and Fractals, **199**, (2025), 116619.

[25] M. Mohammad, I. A. Baba, E. Hincal and F. A. Rihan, *A novel fractional order model for analyzing counterterrorism operations and mitigating extremism*, Decision Analytics Journal, (2025), 100589.

[26] M. Mohammad and A. Trounev, *Computational precision in time fractional PDEs: Euler wavelets and novel numerical techniques*, Partial Differential Equations in Applied Mathematics, **12**, (2024), 100918.

[27] M. Mohammad, *A Tight Wavelet Frames-Based Method for Numerically Solving Fractional Riccati Differential Equations*, Mathematical Methods in the Applied Sciences, (2025).

[28] I. Podlubny, Fractional Differential Equations, Academic Press, 1998.

[29] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in fractional differentiation and applications, **1**(2), (2015), 73-85.

[30] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Thermal Science, **20**(2), (2016), 763-769.

[31] M. Mohammad and M. Saadaoui, *A new fractional derivative extending classical concepts: Theory and applications*, Partial Differential Equations in Applied Mathematics, **11**, (2024), 100889.

[32] F. A. Rihan and M. N. Anwar, *Qualitative analysis of delayed SIR epidemic model with a saturated incidence rate*, International Journal of Differential Equations, **2012**(1), (2012) 408637.

[33] S. Erman, A. Demir, and E. Ozbilge, *Solving inverse non-linear fractional differential equations by generalized Chelyshkov wavelets*, Alexandira Engineering Journal, **66**, (2023), 947-956. https://doi.org/10.1016/j.aej.2022.10.063

[34] S. Behera and S. Sasha Ray, *An efficient numerical method based on Euler wavelets for solving fractional order pantograph Volterra delay-integro-differential equations*, Journal of Compuatational and Applied Mathematics, **406**, (2022), 113825. https://doi.org/10.1016/j.cam.2021.113825

[35] M. Mohammad and A. Trounev, *A new technique for solving neutral delay differential equations based on Euler wavelets*. Complexity, **2022**(1), (2022), 1753992. https://doi.org/10.1155/2022/1753992