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New Convex Function Inequalities and Generalizations of Bernoulli's Inequality

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Abstract: In this paper, we present a generalized equivalence result for convex functions that extends previous characterizations. Our approach provides a simpler and stronger proof by combining classical convexity tools and right-hand derivative analysis, avoiding complex case-by-case arguments. This structure leads to new functional inequalities of Bernoulli type with wider applicability.

Keywords: Bernoulli inequalities, convex functions, inequalities.
AMS Math Codes: 26D10, 26D15, 26A51

1 Introduction

Convex inequalities involving power-type expressions have found broad utility in various areas of analysis and geometry, particularly in the study of approximate orthogonality in normed spaces and the structure of operator spaces. Bernoulli's inequality has long served as a fundamental tool in real analysis and its numerous applications. Motivated by its significance, a considerable number of researchers in recent years have proposed various extensions and generalizations of this classical result. For a more comprehensive understanding of Bernoulli's inequality, the reader is encouraged to consult the detailed discussions and analyses presented in [3], [4], [5], as well as the references cited therein.

A notable result in this context is due to Chmieliński et al. [2], who proved the following elegant characterization:

Theorem 1.1. *[2] Let $f, g : I \to [0, +\infty)$ be convex functions such that $f(0) = 1$ and $g(0) = 0$. Then for any $\alpha \in (1, +\infty)$, the following conditions are equivalent:*

1. $1 \leq f(t) + g(t)$ *for all* $t \in I$;
2. $1 \leq f(t)^{\alpha} + \alpha g(t)$ *for all* $t \in I$.

Applications of Theorem 1.1 are far-reaching. For instance, it has been employed in the study of approximate Birkhoff-James orthogonality and in deriving new geometric inequalities in operator theory [1] and [2]. However, while the result itself is elegantly simple, the original proof is surprisingly lengthy and relies crucially on the classical Bernoulli inequality:

$$1 + \alpha b \leq (1 + b)^{\alpha}, \quad b \in (-1, 0], \quad \alpha > 1.$$

Motivated by the simplicity of the statement and the complexity of the existing proof, we sought an alternative approach. In this paper, we present a new proof technique based on the monotonicity of the right derivative of convex functions—an elementary yet powerful tool that allows us to avoid the Bernoulli inequality entirely. Moreover, we obtain a strictly stronger result by expanding the functional framework:

Theorem 1.2. *Let $I \subseteq [0, +\infty)$ be an interval containing 0, and let $f, b, g : I \to [0, +\infty)$ be convex functions such that $f(0) = b(0) = 1$ and $g(0) = 0$. Then for every $\alpha \in (1, +\infty)$, the following statements are equivalent:*

1. $2 \leq f(t) + b(t) + g(t)$ *for all* $t \in I$;
2. $2 \leq f(t)^\alpha + b(t)^\alpha + \alpha g(t)$ *for all* $t \in I$.

This theorem not only generalizes Theorem 1.1, but also provides a robust framework for generating new functional inequalities. In particular, we demonstrate how this result leads to new and improved versions of Bernoulli-type inequalities and power sum estimates. These inequalities, derived as direct consequences of Theorem 2.1, exhibit the versatility and strength of our approach and open avenues for further developments in convex analysis, operator inequalities, and nonlinear function theory.

2 Generalized Bernoulli-Type Inequalities

Our main result, which generalizes the preceding theorem and forms the foundation for the subsequent applications, is presented below:

Theorem 2.1. *Let $I \subseteq [0, +\infty)$ be an interval containing 0, and let $f, b, g : I \to [0, +\infty)$ be convex functions such that*

$$f(0) = 1, \qquad b(0) = 1, \qquad g(0) = 0.$$

Then, for any $\alpha \in (1, +\infty)$, the following statements are equivalent:

(a) $f(t) + b(t) + g(t) \geq 2$ *for all* $t \in I$;

(b) $f(t)^\alpha + b(t)^\alpha + \alpha g(t) \geq 2$ *for all* $t \in I$.

Proof. Define the auxiliary functions

$$\gamma(t) := f(t) + b(t) + g(t), \qquad \varphi(t) := f(t)^\alpha + b(t)^\alpha + \alpha g(t).$$

(a) $\Rightarrow$ (b)

Assume that $\gamma(t) \geq 2$ for all $t \in I$. Since $x \mapsto x^\alpha$ is convex and increasing on $[0, \infty)$ for $\alpha > 1$, Jensen's inequality gives

$$\frac{f(t)^\alpha + b(t)^\alpha}{2} \geq \left(\frac{f(t) + b(t)}{2} \right)^\alpha.$$

From (a),

$$f(t) + b(t) \geq 2 - g(t).$$

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Thus,

$$f(t)^{\alpha} + b(t)^{\alpha} \geq 2\left(1 - \frac{g(t)}{2}\right)^{\alpha}.$$

Therefore,

$$\varphi(t) = f(t)^{\alpha} + b(t)^{\alpha} + \alpha g(t) \geq 2\left(1 - \frac{g(t)}{2}\right)^{\alpha} + \alpha g(t).$$

Let $x := g(t) \in [0,2]$ (note that $g(t) \geq 0$ and $f(t), b(t) \geq 0$ imply $x \leq 2$). Define

$$h(x) := 2(1 - x/2)^{\alpha} + \alpha x.$$

By taking the derivative:

$$h'(x) = \alpha\big[1 - (1 - x/2)^{\alpha-1}\big].$$

Since $x \in (0,2)$ implies $(1 - x/2) \in (0,1)$ and $\alpha > 1$, we have

$$(1 - x/2)^{\alpha-1} < 1 \quad \Rightarrow \quad h'(x) > 0.$$

Hence h is strictly increasing on $[0,2]$, so

$$h(x) \geq h(0) = 2.$$

Thus,

$$\varphi(t) = h(g(t)) \geq 2,$$

which proves (b).

(b) $\Rightarrow$ (a)

Assume that $\varphi(t) \geq 2$ for all $t \in I$. Since

$$\varphi(0) = f(0)^{\alpha} + b(0)^{\alpha} + \alpha g(0) = 1 + 1 + 0 = 2,$$

the function φ attains a minimum at $t = 0$. Hence the right derivative must satisfy

$$D^{+}\varphi(0) \geq 0.$$

Because f, b, g are convex, the right derivatives $D^{+}f(0)$, $D^{+}b(0)$, $D^{+}g(0)$ exist. Using first–order expansions:

$$f(t) = 1 + D^{+}f(0)t + o(t), \qquad b(t) = 1 + D^{+}b(0)t + o(t), \qquad g(t) = D^{+}g(0)t + o(t).$$

Then

$$f(t)^{\alpha} = 1 + \alpha D^{+}f(0)t + o(t), \qquad b(t)^{\alpha} = 1 + \alpha D^{+}b(0)t + o(t).$$

Substituting into $\varphi(t)$:

$$\varphi(t) = 2 + \alpha t\big(D^{+}f(0) + D^{+}b(0) + D^{+}g(0)\big) + o(t).$$

Thus,

$$D^{+}\varphi(0) = \alpha\big(D^{+}f(0) + D^{+}b(0) + D^{+}g(0)\big) = \alpha D^{+}\gamma(0).$$

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From $D^+\varphi(0) \geq 0$ and $\alpha > 1$ we conclude:

$$D^+\gamma(0) \geq 0.$$

Since γ is convex, satisfies $\gamma(0) = 2$, and has $D^+\gamma(0) \geq 0$, it must be non-decreasing for $t \geq 0$. Therefore,

$$\gamma(t) \geq \gamma(0) = 2 \qquad \text{for all } t \in I.$$

Hence (a) holds. □

Remark 2.2. *Compared to the original proof of related results, which often involves intricate piecewise analysis, the approach presented here offers a unified framework. The implication* $(\mathbf{a}) \Rightarrow (\mathbf{b})$ *is concisely derived using* ***Jensen's inequality*** *and a basic monotonicity argument. Crucially, the implication* $(\mathbf{b}) \Rightarrow (\mathbf{a})$ *is achieved through the analysis of the* ***right-hand derivative*** *at the boundary point,* $D^+\varphi(0)$*. This method, while still rigorous, establishes the full equivalence without the need for delicate case distinctions often required when comparing* x^α *behaviour on* $[0,1]$ *and* $(1,\infty)$*. As such, this proof maintains clarity and highlights the necessary and sufficient role of convexity at the initial point* $t = 0$*.*

Theorem 2.3. *Let* $\alpha > 1$ *and* $x > -1$*. Then the Bernoulli inequality holds:*

$$(1+x)^\alpha \geq 1 + \alpha x.$$

Proof. First, we establish the inequality for the non-negative case using our main result. Let $I = [0, \infty)$ and define the convex functions $f, b, g : I \to [0, \infty)$ as:

$$f(t) := 1, \qquad b(t) := 1 + t, \qquad g(t) := -t.$$

These functions satisfy the initial conditions $f(0) = 1$, $b(0) = 1$, and $g(0) = 0$. Furthermore, for any $t \in I$, their sum is constant:

$$f(t) + b(t) + g(t) = 1 + (1+t) - t = 2.$$

By applying Theorem 2.1, we obtain:

$$f(t)^\alpha + b(t)^\alpha + \alpha g(t) \geq 2 \quad \Rightarrow \quad 1^\alpha + (1+t)^\alpha + \alpha(-t) \geq 2,$$

which simplifies to the Bernoulli inequality for the positive case:

$$(1+t)^\alpha \geq 1 + \alpha t, \qquad \forall t \geq 0. \tag{2.1}$$

Now, we consider the case where $x \in (-1, 0]$. To remain consistent with the domain of Theorem 2.1, let $b \in (-1, 0]$ and define $x = 1 + b$, where $x \in (0, 1]$. The desired inequality $(1+b)^\alpha \geq 1 + \alpha b$ is equivalent to:

$$x^\alpha - \alpha x + \alpha - 1 \geq 0.$$

Define the auxiliary function $\psi : (0, 1] \to \mathbb{R}$ by

$$\psi(x) := x^\alpha - \alpha x + \alpha - 1.$$

We observe that $\psi(1) = 1^\alpha - \alpha(1) + \alpha - 1 = 0$. Taking the derivative with respect to x, we find:

$$\psi'(x) = \alpha x^{\alpha-1} - \alpha = \alpha(x^{\alpha-1} - 1).$$

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Since $x \in (0,1]$ and $\alpha > 1$, it follows that $x^{\alpha-1} \leq 1$, which implies $\psi'(x) \leq 0$. Thus, ψ is a non-increasing function on $(0,1]$. This leads to:

$$\psi(x) \geq \psi(1) = 0, \qquad \forall x \in (0,1].$$

showing it is non-negative and decreasing to zero at x=1] This confirms that the inequality holds for $b \in (-1,0]$. Combining this with (2.1), we conclude that $(1+x)^\alpha \geq 1 + \alpha x$ for all $x > -1$. □

Theorem 2.4. *Let* $a, b, c > 0$ *and let* $\alpha > 1$. *If*

$$a + b + c \geq 2,$$

then

$$a^\alpha + b^\alpha + \alpha c \geq 2.$$

Proof. Define the functions $f, h, g : [0,1] \to [0, +\infty)$ as affine (hence convex) functions by

$$f(t) := 1 + (a-1)t, \quad h(t) := 1 + (b-1)t, \quad g(t) := ct.$$

These functions satisfy the initial conditions

$$f(0) = 1, \quad h(0) = 1, \quad g(0) = 0.$$

Now consider their sum

$$\gamma(t) := f(t) + h(t) + g(t) = 2 + (a + b + c - 2)t.$$

Since the hypothesis $a + b + c \geq 2$ implies $a + b + c - 2 \geq 0$, we obtain

$$\gamma(t) \geq 2, \quad \forall\, t \in [0,1].$$

Hence condition (a) of Theorem 2.1 is satisfied for all $t \in [0,1]$.

By Theorem 2.1, condition (b) follows:

$$f(t)^\alpha + h(t)^\alpha + \alpha g(t) \geq 2, \quad \forall\, t \in [0,1].$$

Evaluating this inequality at $t = 1$, we obtain

$$f(1) = a, \quad h(1) = b, \quad g(1) = c,$$

which yields

$$a^\alpha + b^\alpha + \alpha c \geq 2.$$

This completes the proof. □

Theorem 2.5. *Let* $\alpha > 1$*, and let* $a, b, c \geq 0$ *satisfy*

$$a + b + c = 2.$$

Then, the following inequality holds:

$$(1+a)^\alpha + (1+b)^\alpha + \alpha c \geq 2.$$

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Proof. Define the functions $f, b, g : [0, +\infty) \to [0, +\infty)$ by

$$f(t) := 1 + at, \quad b(t) := 1 + bt, \quad g(t) := ct.$$

To apply Theorem 2.1, we verify that the functions f, b, g satisfy its hypotheses. Each function is affine and hence convex on $[0, +\infty)$. Moreover, they satisfy the initial conditions:

$$f(0) = 1, \quad b(0) = 1, \quad g(0) = 0.$$

Next, consider their sum for arbitrary $t \geq 0$:

$$f(t) + b(t) + g(t) = (1 + at) + (1 + bt) + ct = 2 + (a + b + c)t = 2 + 2t \geq 2,$$

since $a+b+c = 2$. Therefore, the functions f, b, g satisfy all conditions required by Theorem 2.1. Applying Theorem 2.1 yields

$$f(t)^\alpha + b(t)^\alpha + \alpha g(t) \geq 2 \quad \forall t \geq 0.$$

Evaluating this inequality at $t = 1$ completes the proof:

$$(1 + a)^\alpha + (1 + b)^\alpha + \alpha c \geq 2.$$

□

Theorem 2.6. *Let $\alpha > 1$ and let $a, b, c > 0$ satisfy*

$$a + b + c \geq 2.$$

Then the following inequality holds:

$$a^\alpha + b^\alpha + \alpha c \geq 2.$$

Proof. Define the affine (hence convex) functions

$$f(t) = 1 + (a - 1)t, \qquad b(t) = 1 + (b - 1)t, \qquad g(t) = ct, \quad t \in [0, 1].$$

These functions satisfy

$$f(0) = 1, \qquad b(0) = 1, \qquad g(0) = 0.$$

Their sum is

$$f(t) + b(t) + g(t) = 2 + (a + b + c - 2)t.$$

Since $a + b + c \geq 2$, we have

$$f(t) + b(t) + g(t) \geq 2 \quad \text{for all } t \in [0, 1].$$

Thus condition *(a)* of Theorem 2.1 is satisfied. By Theorem 2.1, condition *(b)* follows:

$$f(t)^\alpha + b(t)^\alpha + \alpha g(t) \geq 2 \quad \text{for all } t \in [0, 1].$$

Evaluating at $t = 1$ gives

$$a^\alpha + b^\alpha + \alpha c \geq 2,$$

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which completes the proof. □

Conclusion

We have presented a generalization of the classical Bernoulli-type equivalence, establishing a new characterization for convex functions involving power expressions (Theorem 2.1). Our main result demonstrates the equivalence between the linear condition $\mathbf{2} \leq \mathbf{f}(\mathbf{t}) + \mathbf{b}(\mathbf{t}) + \mathbf{g}(\mathbf{t})$ and the power condition $\mathbf{2} \leq \mathbf{f}(\mathbf{t})^{\alpha} + \mathbf{b}(\mathbf{t})^{\alpha} + \alpha\mathbf{g}(\mathbf{t})$.

This work introduces a **methodologically robust** approach to proving this equivalence. Crucially, the implication $(\mathbf{b}) \Rightarrow (\mathbf{a})$ is rigorously established by analyzing the **right-hand derivative** of the auxiliary function at the initial point, $t = 0$. This technique provides a clear and unified alternative to existing proofs, which often rely on intricate piecewise case analysis and specialized inequalities.

The versatility of Theorem 2.1 was demonstrated through its application to various functional inequalities. Specifically, we provided a concise re-derivation of the classical Bernoulli inequality and established new power-sum estimates using **affine constructions**. These applications confirm that the proposed framework not only simplifies existing results but also extends the utility of convex analysis techniques.

Future research may explore **multivariate generalizations** involving more than three functions, development of **integral-type inequalities**, and applications in operator theory or the geometry of normed spaces, where such power-type estimates are fundamental.

Declarations

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