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A Discrete-time Mathematical Model of Smoking Dynamics with Two Sub-populations of Smokers

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Abstract: We analyze smoking dynamics with two sub-populations of smokers using a discrete-time mathematical model with a standard difference scheme. We divide the smokers' populations into beginners and heavy smokers. The boundedness of the solution is obtained. The equilibrium stability is assessed through the Jury stability conditions. We additionally present numerical simulations to validate the analytical results by giving several examples to depict the stability of all equilibriums. The sensitivity analysis of the model's parameters is performed to that can offer recommendations for regulators to reduce the number of smokers.

Keywords: Smoking; Standard difference scheme; Boundedness; Jury stability conditions; Sensitivity analysis.
AMS Math Codes: 39A60; 92D30.

1. Introduction

Smoking continues to be one of the leading causes of preventable diseases and deaths worldwide, with significant implications for public health and socioeconomic well-being [1]. The World Health Organization (WHO) reports that tobacco use causes over 8 million deaths annually, with more than 80 % of smokers residing in low- and middle-income countries where the burden of smoking-related illnesses is disproportionately high [2]. Smoking is strongly associated with a range of serious health conditions, particularly lung cancer [3]. Despite decades of public health campaigns, regulatory policies, and smoking cessation initiatives, the prevalence of smoking remains alarmingly high in many regions, driven by factors such as nicotine addiction, cultural acceptance, and aggressive marketing by tobacco companies [4, 1]. Smoking cigarettes and quitting are complicated behaviors influenced by a number of circumstances [5].

Mathematical modeling has become a powerful tool for analyzing the complex dynamics of smoking within populations. Various studies have explored different aspects of smoking behavior and control strategies using mathematical frameworks. Zhang et al. (2020) investigated a delayed quitting smoking model incorporating a harmonic mean type and optimal legislative control strategies to minimize the number of smokers [6]. Similarly, Sofia et al. (2023) developed a nonlinear mathematical model to examine the impact of media awareness on reducing smoking transmission from smokers to non-smokers [7]. Herdiana et al. (2022) proposed a mathematical model that highlights the effectiveness of combined therapy-educational campaigns, counseling, and nicotine replacement therapy-in reducing smoking prevalence in mixed populations of heavy and beginner smokers [8]. A simplified version of this model was further analyzed in a subsequent study [9]. Zhang et al. (2024) explored a smoking epidemic model in both deterministic

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and stochastic settings, demonstrating that effective smoking control requires accurate tracking of initial smoking population sizes and the implementation of efficient intervention measures [10]. Han et al. (2025) extended this work by applying the Black-Karasinski process to study the global dynamics of a stochastic smoking epidemic model [11]. Recent studies have also incorporated fractional mathematical models to improve long-term forecasting of smoking trends, offering a more nuanced understanding of smoking dynamics over time [12, 13]. These studies highlight the growing role of mathematical modeling in understanding smoking dynamics and designing effective public health strategies.

Mathematical models enable researchers to simulate smoking-related behaviors. Continuous-time differential equations are used in the majority of mathematical models of smoking dynamics as mentioned above. However, in actual situations, smoking behavior is frequently impacted by isolated incidents (e.g., policy changes, awareness campaigns, and peer influence). Because it can more accurately depict these real-world, stepwise behavioral changes, a discrete-time model is required. Discrete-time models, in particular, are effective in capturing the temporal evolution of smoking dynamics over defined time periods. In this study, we propose a discrete-time mathematical model of smoking dynamics based on the continuous model in [8, 9]. The model considers an interaction between beginners and heavy smokers. By examining the stability and sensitivity of the model, we aim to provide insights into the mechanisms driving the smoking prevalence and offer recommendations for mitigating the smoking epidemic.

2. The model and the boundedness of its solution

Ansori and Herdiana [9] studied the smoking dynamics of active smokers in a mixed population via the following system of differential equations:

$$\begin{cases} \dfrac{dP}{dt} = \Lambda - (\alpha B + \beta S)P - \mu P + \sigma B \\ \dfrac{dB}{dt} = (\alpha B + \beta S)P - \delta BS - (\sigma + \mu)B. \\ \dfrac{dS}{dt} = \delta BS - \mu S \end{cases} \tag{2.1}$$

All parameters are positive.

The explanation of the parameters is as follows. Parameter Λ (constant growth rate of potential smokers) represents the rate at which new individuals enter the population as potential smokers. This could be due to birth rates, migration, or individuals reaching an age where they are susceptible to smoking influences. Parameter α (effective interaction rate between potential smokers and beginner smokers) measures how frequently potential smokers interact with beginner smokers in a way that encourages them to start smoking. This could be through peer influence, school or workplace environments, or social settings where smoking is normalized. Parameter β (effective interaction rate between potential smokers and heavy smokers) represents the rate at which potential smokers are influenced by heavy smokers to start smoking. This includes influences such as family members who smoke, media exposure, or cultural acceptance of smoking. Parameter δ (effective interaction rate between beginner smokers and heavy smokers) denotes the likelihood of beginner smokers progressing to heavy smoking due to continued exposure to heavy smokers. This transition can be driven by increased nicotine dependence, social reinforcement, or lack of smoking cessation support. Parameter σ (self-control smoking-quit rate for beginner smokers) reflects the rate at which beginner smokers successfully quit smoking on their own and return to the potential smoker population. Factors influencing this include personal motivation, education, awareness campaigns, or short-term health concerns. Parameter μ (natural death rate for all populations) represents the general mortality rate affecting all individuals in the model, independent of smoking. This accounts for deaths due to aging, diseases unrelated to smoking, accidents, or other natural causes.

Here, we consider a discrete form of model (2.1) by using a standard difference scheme (Euler's forward method) as follows:

$$\begin{cases} P_{t+1} = P_t + h[\Lambda - (\alpha B_t + \beta S_t)P_t - \mu P_t + \sigma B_t] \\ B_{t+1} = B_t + h[(\alpha B_t + \beta S_t)P_t - \delta B_t S_t - (\sigma + \mu)B_t], \\ S_{t+1} = S_t + h[\delta B_t S_t - \mu S_t] \end{cases} \tag{2.2}$$

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where $h > 0$ and the initial conditions $P_0, B_0, S_0 \geq 0$.

Due to various biological and behavioral aspects of smoking dynamics, we took into account the mass action incidence rate instead of the standard incidence rate when modeling the spread of smoking behavior as an infectious-like process. Direct social interactions, such as peer pressure and influence from family members or close social circles, are frequently the cause of smoking initiation. The mass action incidence rate makes the assumption that the number of interactions between susceptible people and existing smokers is proportionate to the rate of new smokers. The standard incidence rate introduces a sort of "saturation effect.[a]s prevalence rises and is frequently employed in epidemiological models where the risk of infection is based on the percentage of infected people in the overall population. The risk of initiation does not, however, always reach saturation in smoking dynamics. People are exposed to smokers in a variety of contexts (such as the media, workplaces, and schools), and the likelihood that someone will start smoking depends more on absolute numbers than on population proportions. Many epidemiological models for behaviors affected by social interactions-e.g. drug use [14] and alcohol addiction [15]-adopt the mass action form since direct exposure to current users raises the likelihood of adopting such behaviors. Unlike traditional infectious diseases like measles [16], influenza [17], hepatitis B [18], or COVID-19 [19], which are spread by relative population sizes, smoking behaves similarly to such social behaviors.

The population in our model is assumed to be homogeneous, meaning that each person has an equal chance of interacting, changing, or reacting to processes (e.g., infection, quitting smoking, reproduction, death, etc.). Mass action mixing occurs when people mix evenly with one another. Everyone changes states (such as being susceptible to infection) at the same rate. There's no heterogeneity in traits like age, immunity, risk behavior, etc.

First, we study the boundedness of the system's solution. We follow the method used in [20, 21]. Let $N_t = P_t + B_t + S_t$ be the total number of population at time t. The initial condition yields $N_0 = P_0 + B_0 + S_0 \geq 0$. By summing up all equations in (2.2) we have

$$N_{t+1} = N_t + h(\Lambda - \mu N_t) = h\Lambda - (1 - h\mu)N_t.$$

This is a linear difference equation. By using standard technique (finding the homogeneous and particular solutions), we get the solution is as follows

$$N_t = \left(N_0 - \frac{\Lambda}{\mu}\right)(1 - h\mu)^t + \frac{\Lambda}{\mu}. \tag{2.3}$$

To make the solution N_t does not diverge, the absolute value of $1 - h\mu$ should less than 1.

Next, we prove that the solution of system (2.2) is bounded. The statement is given in the following theorem.

Theorem 2.1. *Let* $0 < h \leq 1/\left(\frac{(\alpha+\beta+\delta)\Lambda}{\mu} + \sigma + \mu\right)$ *and* $N_0 \leq \Lambda/\mu$. *Then the solution of system* (2.2) *is always nonnegative and bounded above.*

Demostración. First, we analyze the behavior of the solution N_t from (2.3). Let $0 < h < 1/\left(\frac{(\alpha+\beta+\delta)\Lambda}{\mu} + \sigma + \mu\right)$. Then $h < 1/\mu$. This implies $|1 - h\mu| < 1$. Therefore, we have $N_t \to \Lambda/\mu$ as $t \to \infty$. Let $N_0 \leq \Lambda/\mu$, then N_t is an increasing function of t and $N_t \leq \Lambda/\mu$. This means that N_t is bounded above. Since $P_t \leq N_t \leq \Lambda/\mu$, $B_t \leq N_t \leq \Lambda/\mu$, and $S_t \leq N_t \leq \Lambda/\mu$, then the solution of system (2.2) is bounded above.

Next, observe that

$$h \leq \frac{1}{\frac{(\alpha+\beta+\delta)\Lambda}{\mu} + \sigma + \mu} = \frac{1}{\alpha\frac{\Lambda}{\mu} + (\beta+\delta)\frac{\Lambda}{\mu} + \sigma + \mu} \leq \frac{1}{\alpha B_t + (\beta+\delta)S_t + \sigma + \mu}.$$

From this, we have three inequalities:

$$\begin{aligned}
(i) \quad & h \leq \frac{1}{\alpha B_t + (\beta+\delta)S_t + \sigma + \mu} < \frac{1}{\alpha B_t + \beta S_t + \mu},\\
(ii) \quad & h \leq \frac{1}{\alpha B_t + (\beta+\delta)S_t + \sigma + \mu} < \frac{1}{\delta S_t + \sigma + \mu},\\
(iii) \quad & h \leq \frac{1}{\alpha B_t + (\beta+\delta)S_t + \sigma + \mu} < \frac{1}{\mu}.
\end{aligned}$$

To prove that the solution of system (2.2) is nonnegative, or in other words, $P_t, B_t, S_t \geq 0$, for all $t \geq 0$, we use mathematical induction. First, assume that $P_t, B_t, S_t \geq 0$, for $t \geq 0$. We will prove that $P_{t+1}, B_{t+1}, S_{t+1} \geq 0$. From the first equation in (2.2), we get

$$P_{t+1} \geq P_t - h(\alpha B_t + \beta S_t + \mu)P_t = [1 - h(\alpha B_t + \beta S_t + \mu)]P_t.$$

From (i), we have $1 - h(\alpha B_t + \beta S_t + \mu) \geq 0$. This implies $P_{t+1} \geq 0$.

From the second equation in (2.2), we get

$$B_{t+1} \geq B_t - h(\delta S_t + \sigma + \mu)B_t = [1 - h(\delta S_t + \sigma + \mu)]B_t.$$

From (ii), we have $1 - h(\delta S_t + \sigma + \mu) \geq 0$. This implies $B_{t+1} \geq 0$.

From the third equation in (2.2), we get

$$S_{t+1} \geq S_t - h\mu B_t = [1 - h\mu]B_t.$$

From (iii), we have $1 - h\mu \geq 0$. This implies $S_{t+1} \geq 0$.

This completes the proof. □

3. The local stability of the equilibriums

The stability analysis of equilibrium points of (2.2) is essential for comprehending the smoking population's long-term behavior. Stability analysis aids in forecasting whether smoking will continue or eventually decline in the general population. It suggests that smoking can be completely eliminated under specific circumstances if the smoker-free equilibrium remains steady. By ensuring that the model behaves realistically and maintains its robustness under various circumstances, stability analysis increases the credibility of the results for policymakers.

The model (2.2) has four equilibrium points, namely $E_i = (P, B, S)$, $i = 0, 1, 2$. First, the smoking-free equilibrium is obtained by setting $B = S = 0$ as follows

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right).$$

Second, the heavy smokers-free equilibrium is obtained by setting $S = 0$, which given by

$$E_1 = \left(\frac{\sigma + \mu}{\alpha}, \frac{\Lambda\alpha - \mu(\sigma + \mu)}{\alpha\mu}, 0\right).$$

Next, the positive equilibrium ($P, B, S > 0$) is obtained as follows. From the third equation in (2.2), we have $0 = hS(\delta B - \mu)$, or $B = \mu/\delta$. If we substitute this into the first and second equations in (2.2) and eliminate them, then we obtain

$$P = \left(\frac{\Lambda}{\mu} - \frac{\mu}{\delta}\right) - S,$$

and

$$0 = \beta S^2 + \left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right) S + \left(\frac{(\sigma+\mu)\mu - \alpha\Lambda}{\delta} + \frac{\alpha\mu^2}{\delta^2}\right).$$

The solution of the quadratic equation is as follows

$$S_{1,2} = \frac{-\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right) \pm \sqrt{\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right)^2 - 4\beta\left(\frac{(\sigma+\mu)\mu - \alpha\Lambda}{\delta} + \frac{\alpha\mu^2}{\delta^2}\right)}}{2\beta}.$$

Therefore, we obtain two equilibriums:

$$E_2 = \left(P_1^*, \frac{\mu}{\delta}, S_1^*\right),$$
$$E_3 = \left(P_2^*, \frac{\mu}{\delta}, S_2^*\right),$$

where

$$S_1^* = \frac{-\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right) + \sqrt{\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right)^2 - 4\beta\left(\frac{(\sigma+\mu)\mu - \alpha\Lambda}{\delta} + \frac{\alpha\mu^2}{\delta^2}\right)}}{2\beta},$$
$$P_1^* = \left(\frac{\Lambda}{\mu} - \frac{\mu}{\delta}\right) - S_1^*,$$
$$S_2^* = \frac{-\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right) - \sqrt{\left(\mu + \frac{(\alpha+\beta)\mu}{\delta} - \frac{\beta\Lambda}{\mu}\right)^2 - 4\beta\left(\frac{(\sigma+\mu)\mu - \alpha\Lambda}{\delta} + \frac{\alpha\mu^2}{\delta^2}\right)}}{2\beta},$$
$$P_2^* = \left(\frac{\Lambda}{\mu} - \frac{\mu}{\delta}\right) - S_2^*.$$

Note that, the equilibrium point E_1 exists if $\frac{\Lambda\alpha - \mu(\sigma+\mu)}{\alpha\mu} > 0$, or

$$R_0 := \frac{\Lambda\alpha}{\mu(\sigma+\mu)} > 1.$$

In [9], $R_0 = \frac{\Lambda\alpha}{\mu(\sigma+\mu)}$ is the basic reproduction number. In the context of the smoking model, R_0 is the average number of new smokers produced by a single smoker in a population that is fully susceptible over a specific time period. The prevalence of smoking will gradually decrease if $R_0 < 1$, meaning that each smoker contributes to fewer than one new smoker. This suggests that smoking will eventually become less common. A sustained or expanding smoking population results from each smoker contributing to multiple new smokers if $R_0 > 1$.

The Jacobian matrix of system (2.2) evaluated at any point $E = (P, B, S)$ is given by

$$J(E) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix}, \tag{3.1}$$

where

$$j_{11} = 1 - h[(\alpha B + \beta S) + \mu], \quad j_{12} = h[-\alpha P + \sigma], \quad j_{13} = -h\beta P,$$
$$j_{21} = h[\alpha B + \beta S], \quad j_{22} = 1 + h[\alpha P - \delta S - (\sigma + \mu)], \quad j_{23} = h[\beta P - \delta B],$$
$$j_{32} = h\delta S, \quad j_{33} = 1 + h[\delta B - \mu].$$

The characteristic of the Jacobian matrix (3.1) is provided below

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

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where

$$\begin{aligned}
a_1 &= -(j_{11} + j_{22} + j_{33}), \\
a_2 &= j_{11}(j_{22} + j_{33}) + j_{22}j_{33} - j_{32}j_{23} - j_{21}j_{12}, \\
a_3 &= j_{21}(j_{12}j_{33} - j_{32}j_{13}) - j_{11}(j_{22}j_{33} - j_{32}j_{23}).
\end{aligned}$$

We use the Jury stability conditions to analyze the local stability of system (2.2) which is given below [22]:

$$\begin{cases} C_1 := 1 + a_1 + a_2 + a_3 > 0 \\ C_2 := 1 - a_1 + a_2 - a_3 > 0 \\ C_3 := 1 - a_2 + a_1a_3 - a_3^2 > 0 \\ C_4 := 3 - a_2 > 0. \end{cases} \tag{3.2}$$

The equilibrium is said to be locally asymptotically stable if the condition (3.2) is satisfied.

First, we study the conditions for the local stability of equilibrium E_0. For this case, we have the Jacobian matrix of system (2.2) evaluated at E_0 below

$$J(E_0) = \begin{bmatrix} 1 - h\mu & h(-\frac{\alpha\Lambda}{\mu} + \sigma) & -h\frac{\beta\Lambda}{\mu} \\ 0 & 1 + h(\frac{\alpha\Lambda}{\mu} - (\sigma + \mu)) & h\frac{\beta\Lambda}{\mu} \\ 0 & 0 & 1 - h\mu \end{bmatrix}.$$

From this, we have the Jury stability conditions for equilibrium E_0 as follows

$$\begin{aligned}
C_1 = {} & 2h\mu + (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1) + (h\mu - 1)(h\mu \\
& + h(\mu + \sigma - (\Lambda\alpha)/\mu) - 2) + (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1)^2 \\
& + h(\mu + \sigma - (\Lambda\alpha)/\mu) - 2 > 0,
\end{aligned} \tag{3.3a}$$

$$\begin{aligned}
C_2 = {} & (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1) - 2h\mu + (h\mu - 1)(h\mu + h(\mu \\
& + \sigma - (\Lambda\alpha)/\mu) - 2) - (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1)^2 \\
& - h(\mu + \sigma - (\Lambda\alpha)/\mu) + 4 > 0,
\end{aligned} \tag{3.3b}$$

$$\begin{aligned}
C_3 = {} & (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1)^2(2h\mu + h(\mu + \sigma - (\Lambda\alpha)/\mu) \\
& - 3) - (h\mu - 1)(h\mu + h(\mu + \sigma - (\Lambda\alpha)/\mu) - 2) - (h(\mu + \sigma \\
& - (\Lambda\alpha)/\mu) - 1)^2(h\mu - 1)^4 - (h(\mu + \sigma - (\Lambda\alpha)/\mu) - 1)(h\mu - 1) \\
& + 1 > 0
\end{aligned} \tag{3.3c}$$

$$\begin{aligned}
C_4 = {} & 3 - (h\mu - 1)(h\mu + h(\mu + \sigma - (\Lambda\alpha)/\mu) - 2) - (h(\mu + \sigma \\
& - (\Lambda\alpha)/\mu) - 1)(h\mu - 1) > 0.
\end{aligned} \tag{3.3d}$$

Thus, equilibrium E_0 is locally asymptotically stable if the condition (3.3) holds.

For the case of equilibrium E_1, we have the Jacobian matrix as follows

$$J(E_1) = \begin{bmatrix} 1 - h(\mu - \frac{\mu(\mu+\sigma)-\Lambda\alpha}{\mu}) & -h\mu & -\frac{\beta h(\mu+\sigma)}{\alpha} \\ -\frac{h(\mu(\mu+\sigma)-\Lambda\alpha)}{\mu} & 1 & h(\frac{\beta(\mu+\sigma)}{\alpha} + \frac{\delta(\mu(\mu+\sigma)-\Lambda\alpha)}{\alpha\mu}) \\ 0 & 0 & 1 - h(\mu + \frac{\delta(\mu(\mu+\sigma)-\Lambda\alpha)}{\alpha\mu}). \end{bmatrix}$$

This leads us to the following Jury conditions:

$$\begin{aligned}
C_1 = {} & h(\mu - (\mu(\mu + \sigma) - \Lambda\alpha)/\mu) - h^2(\mu(\mu + \sigma) - \Lambda\alpha) \\
& - (h(\mu - (\mu(\mu + \sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu + \sigma)
\end{aligned} \tag{3.4a}$$

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$$- \Lambda\alpha))/(\alpha\mu)) - 1) + (h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 2) - h^2(\mu(\mu+\sigma) - \Lambda\alpha)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1) - 1 > 0$$

$$C_2 = (h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1) - 2h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - h^2(\mu(\mu+\sigma) - \Lambda\alpha) - h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) + (h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 2) + h^2(\mu(\mu+\sigma) - \Lambda\alpha)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1) + 5 > 0 \tag{3.4b}$$

$$C_3 = h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) + h^2(\mu(\mu+\sigma) - \Lambda\alpha) - (h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 2) - ((h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1) + h^2(\mu(\mu+\sigma) - \Lambda\alpha) \times (h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1))(h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) + h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 3) - ((h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1) + h^2(\mu(\mu+\sigma) - \Lambda\alpha)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 1))^2 > 0 \tag{3.4c}$$

$$C_4 = h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) + h^2(\mu(\mu+\sigma) - \Lambda\alpha) - (h(\mu - (\mu(\mu+\sigma) - \Lambda\alpha)/\mu) - 1)(h(\mu + (\delta(\mu(\mu+\sigma) - \Lambda\alpha))/(\alpha\mu)) - 2) + 2 > 0. \tag{3.4d}$$

We will have the equilibrium E_1 is locally asymptotically stable if the condition (3.4) holds.

For the case of equilibriums E_2 and E_3, the expression of the Jury stability conditions C_1, C_2, C_3, and C_4 is too long. Therefore, in this paper, we only calculate them numerically in the examples in the next section. In this section, we only calculate the Jacobian matrices for both equilibriums.

For the third equilibrium, E_2, we have the Jacobian matrix:

$$J(E_2) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & 1 \end{bmatrix}, \tag{3.5}$$

where

$$j_{11} = 1 - h((3\mu)/2 + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2}/2 + (\mu(\alpha+\beta))/(2\delta) - (\Lambda\beta)/(2\mu) + (\alpha\mu)/\delta),$$

$$j_{12} = h(\sigma + \alpha((\mu + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta)),$$

$$j_{13} = \beta h((\mu + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta),$$

$$j_{21} = h(\mu/2 + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta$$

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$$\begin{aligned}
&+ (\alpha\mu^2)/\delta^2))^{1/2}/2 + (\mu(\alpha+\beta))/(2\delta) - (\Lambda\beta)/(2\mu) + (\alpha\mu)/\delta),\\
j_{22} &= 1 - h(\mu + \sigma + \alpha((\mu + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2\\
&- 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta\\
&- (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta) + (\delta(\mu + ((\mu + (\mu(\alpha+\beta))/\delta\\
&- (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2}\\
&+ (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu))/(2\beta)),\\
j_{23} &= -h(\mu + \beta((\mu + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma)\\
&- \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta)\\
&- \Lambda/\mu + \mu/\delta)),\\
j_{32} &= (\delta h(\mu + ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta\\
&+ (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu))/(2\beta).
\end{aligned}$$

In the case of fourth equilibrium E_3, we have the Jacobian matrix:

$$J(E_3) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & 1 \end{bmatrix}, \tag{3.6}$$

where

$$\begin{aligned}
j_{11} &= 1 - h((3\mu)/2 - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma)\\
&- \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2}/2 + (\mu(\alpha+\beta))/(2\delta) - (\Lambda\beta)/(2\mu) + (\alpha\mu)/\delta),\\
j_{12} &= h(\sigma + \alpha((\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma)\\
&- \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu\\
&+ \mu/\delta)),\\
j_{13} &= \beta h((\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta\\
&+ (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta),\\
j_{21} &= h(\mu/2 - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta\\
&+ (\alpha\mu^2)/\delta^2))^{1/2}/2 + (\mu(\alpha+\beta))/(2\delta) - (\Lambda\beta)/(2\mu) + (\alpha\mu)/\delta),\\
j_{22} &= 1 - h(\mu + \sigma + \alpha((\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma)\\
&- \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta)\\
&+ (\delta(\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta\\
&+ (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu))/(2\beta)),\\
j_{23} &= -h(\mu + \beta((\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma)\\
&- \Lambda\alpha)/\delta + (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)/(2\beta) - \Lambda/\mu + \mu/\delta)),\\
j_{32} &= (\delta h(\mu - ((\mu + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu)^2 - 4\beta((\mu(\mu+\sigma) - \Lambda\alpha)/\delta\\
&+ (\alpha\mu^2)/\delta^2))^{1/2} + (\mu(\alpha+\beta))/\delta - (\Lambda\beta)/\mu))/(2\beta).
\end{aligned}$$

4. Numerical simulations

In this section, we give several examples to confirm the analytical result of the previous section, and provide some sensitivity analysis for the parameters. The following initial conditions are used: $P_0 = 153$, $B_0 = 40$, and $S_0 = 79$ [9]. Since the model is formulated in discrete time, solutions are obtained through direct iteration rather than solving differential equations numerically like in continuous-time models that require numerical integration methods. Each iteration step updates the system based on the recurrence relations defined in the model equations. Iterative computation is straightforward and efficient for exploring the system's dynamics over time.

4.1. Illustrative cases for verifying equilibrium stability

Some parameters' values are taken from [9], and the others are assumed such that they satisfy the Jury conditions in (3.2).

Example 4.1. *In this example, we confirm the stability of the smoking-free equilibrium E_0. Consider the value of the following parameters:* $\Lambda = 0{,}25$, $\alpha = 0{,}00014$, $\beta = 0{,}0024$, $\sigma = 0{,}0001$, $\delta = 0{,}0004$, $\mu = 0{,}01$, $h = 0{,}05$. *From* (3.3), *we have* $C_1 = 8{,}25 \times 10^{-11} > 0$, $C_2 = 7{,}99 > 0$, $C_3 = 6{,}88 \times 10^{-10} > 0$, *and* $C_4 = 0{,}0027 > 0$. *Therefore, the equilibrium* $E_0 = (25, 0, 0)$ *is locally asymptotically stable. This result is confirmed by Fig. 1.*

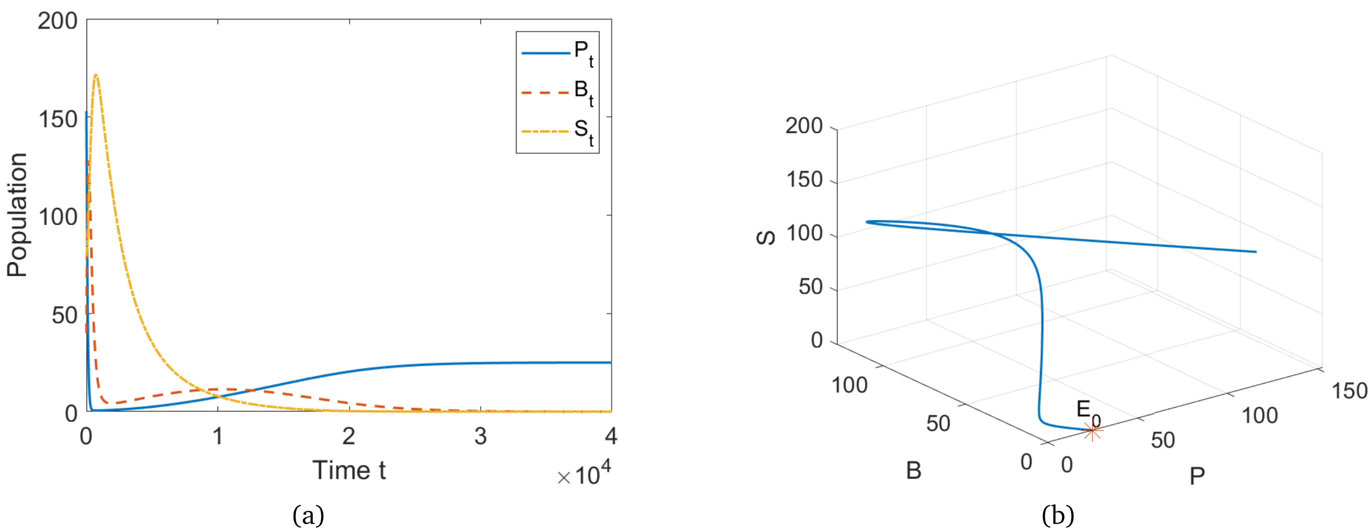

Figura 1: (a) Time series and (b) phase portraits of system (2.2) to depict the the stability of equilibrium E_0.

Example 4.2. *In this example, we confirm the stability of the heavy smokers-free equilibrium E_1. Consider the value of the following parameters:* $\Lambda = 0{,}25$, $\alpha = 0{,}0014$, $\beta = 0{,}0024$, $\sigma = 0{,}0001$, $\delta = 0{,}0004$, $\mu = 0{,}01$, $h = 0{,}05$. *From* (3.4), *we have* $C_1 = 8{,}98 \times 10^{-11} > 0$, $C_2 = 7{,}99 > 0$, $C_3 = 1{,}56 \times 10^{-9} > 0$, *and* $C_4 = 0{,}0038 > 0$. *Therefore, the equilibrium* $E_1 = (7{,}21, 17{,}78, 0)$ *is locally asymptotically stable. This result is confirmed by Fig. 2.*

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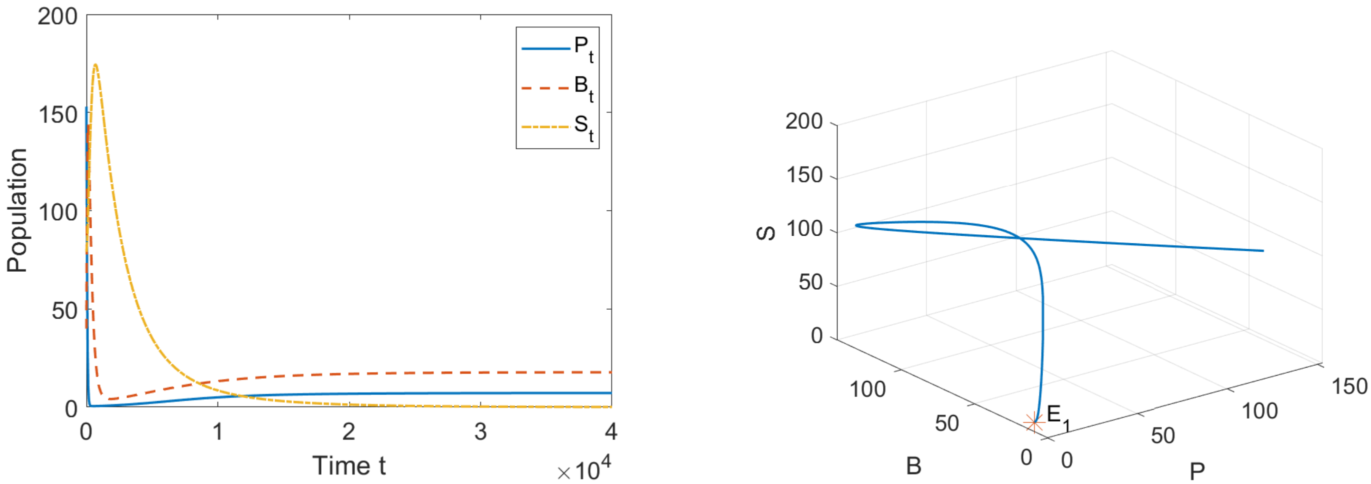
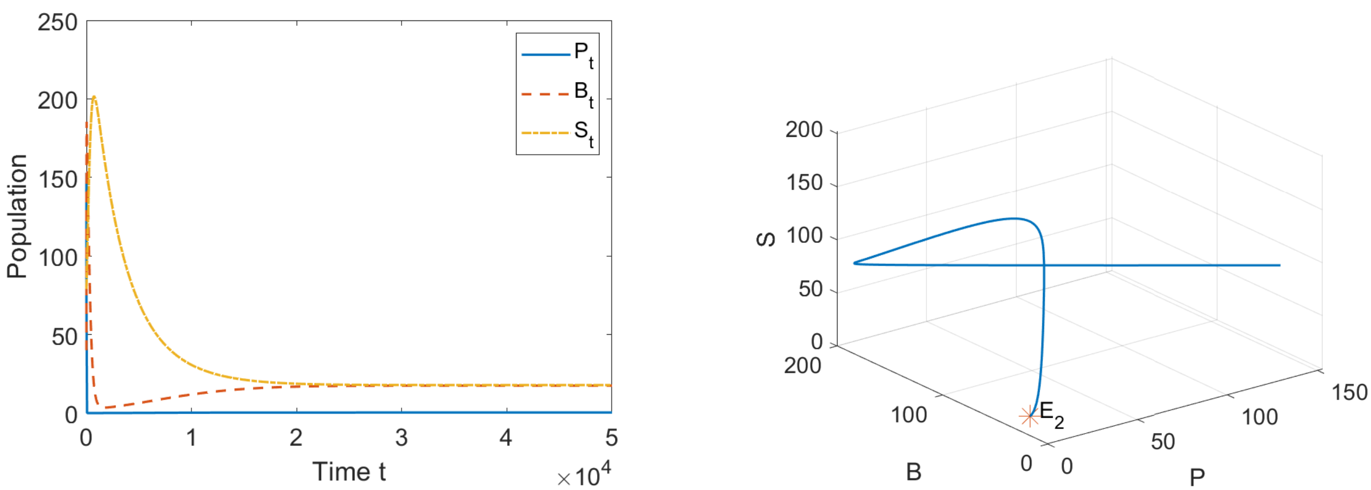


(a) (b)

Figura 2: (a) Time series and (b) phase portraits of system (2.2) to depict the the stability of equilibrium E_1.

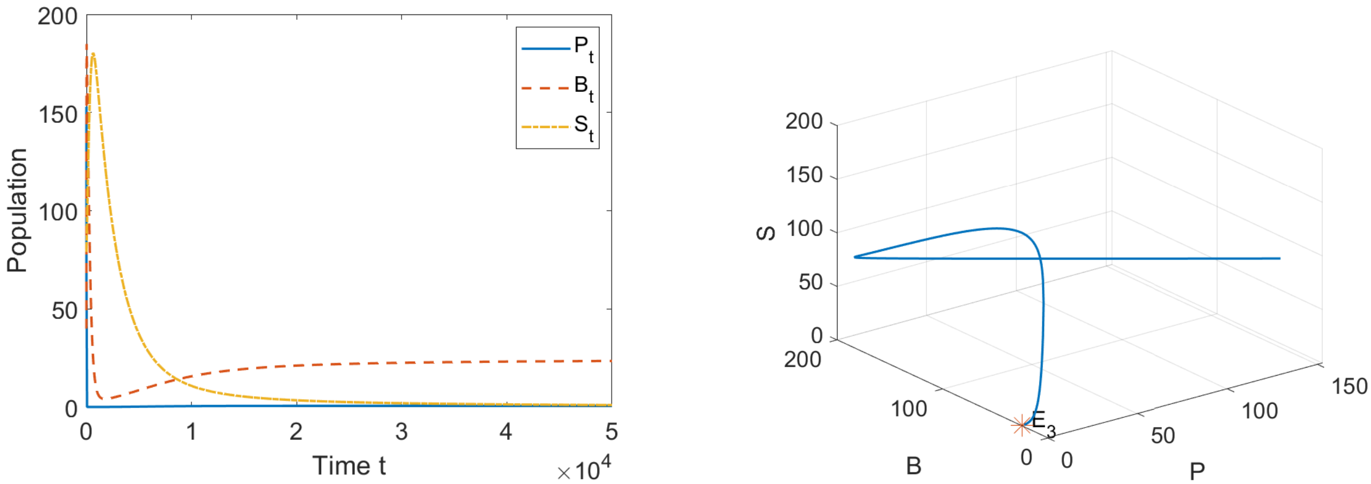
Example 4.3. *In this example, we confirm the stability of the first positive equilibrium* E_2*. Consider the value of the following parameters:* $\Lambda = 0{,}25$*,* $\alpha = 0{,}014$*,* $\beta = 0{,}024$*,* $\sigma = 0{,}0001$*,* $\delta = 0{,}0004$*,* $\mu = 0{,}007$*,* $h = 0{,}05$*. We have the Jury stability conditions:* $C_1 = 1{,}07 \times 10^{-9} > 0$*,* $C_2 = 7{,}92 > 0$*,* $C_3 = 1{,}91 \times 10^{-7} > 0$*, and* $C_4 = 0{,}0393 > 0$*. Therefore, the equilibrium* $E_2 = (8{,}08, 17{,}5, 10{,}13)$ *is locally asymptotically stable. This result is confirmed in Fig. 3.*



(a) (b)

Figura 3: (a) Time series and (b) phase portraits of system (2.2) to depict the the stability of equilibrium E_2.

Example 4.4. *In this example, we confirm the stability of the second positive equilibrium* E_3*. Consider the value of the following parameters:* $\Lambda = 0{,}25$*,* $\alpha = 0{,}014$*,* $\beta = 0{,}024$*,* $\sigma = 0{,}0001$*,* $\delta = 0{,}0004$*,* $\mu = 0{,}0099$*,* $h = 0{,}05$*. We have the Jury stability conditions:* $C_1 = 3{,}79 \times 10^{-11} > 0$*,* $C_2 = 7{,}93 > 0$*,* $C_3 = 1{,}64 \times 10^{-7} > 0$*, and* $C_4 = 0{,}0368 > 0$*. Therefore, the equilibrium* $E_3 = (0{,}29, 24{,}75, 0{,}22)$ *is locally asymptotically stable. This result is confirmed in Fig. 4.*



(a) (b)

Figura 4: (a) Time series and (b) phase portraits of system (2.2) to depict the the stability of equilibrium E_3.

4.2. Sensitivity analysis

The main parameters of system (2.2) which related to smoking behaviors are α (effective interaction rates between potential and beginner smokers), β (effective interaction rates between potential and heavy smokers), δ (effective interaction rates between beginner smokers and heavy smokers), and σ (self-control of smoking rate). In this section, we provide sensitivity analysis to assess the impact of variational of those parameters on the smoking dynamics. To this end, we simulate the system (2.2) with four different values of those parameters. We use the parameters' value based on Example 4.4.

In Figs. 5 and 6, the parameters α and β have a similar sensitivity, where their higher value makes the population of potential smokers, beginners, and heavy smokers decrease, increase, and increase, respectively. Thus, minimizing the interaction between potential smokers and beginners and heavy smokers will reduce the number of smokers.

In Fig. 7, the parameter δ has a very small effect on the potential smoker population. Meanwhile, a higher value of parameter δ makes the beginners and heavy smokers decrease and increase, respectively, with significant impacts. Thus, minimizing the interaction between beginners and heavy smokers can also reduce the potential of heavy smokers.

In Fig. 8, a higher value of parameter σ makes the potential smokers, beginners, and heavy smokers increase, decrease, and decrease, respectively. Thus, maximizing the self-control of smoking will reduce both the beginner and heavy smoker populations. Increasing the number of quit attempts and the success rates of cessation are significant objectives that both call for knowledge of self-control and motivation [23].

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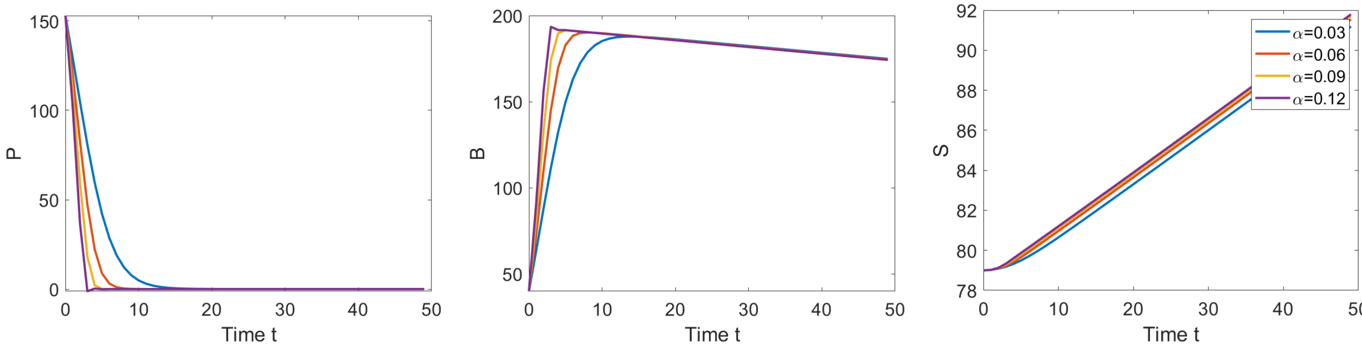
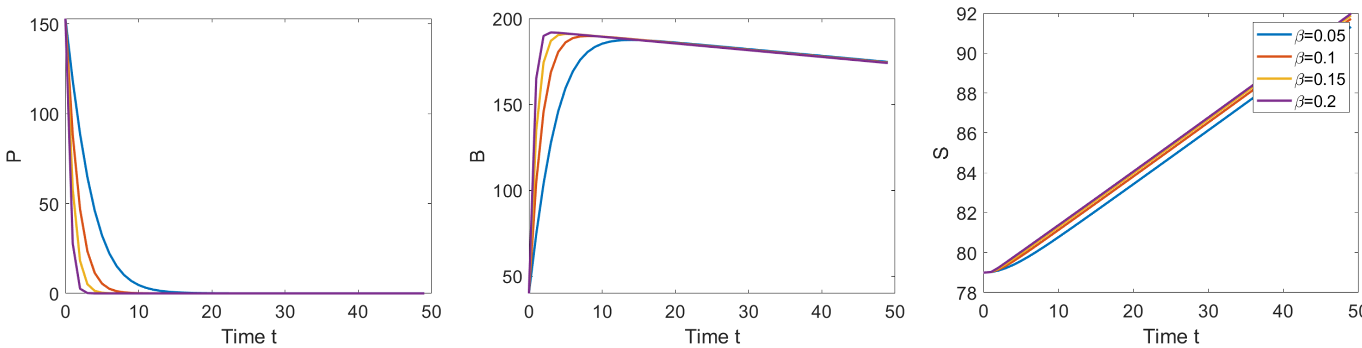


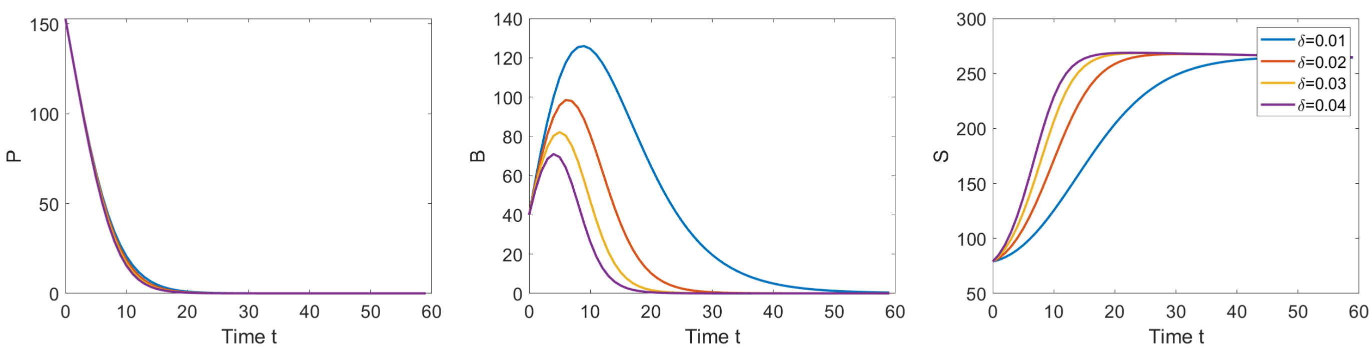
(a) (b) (c)

Figura 5: Sensitivity analysis of parameter α for population (a) potential, (b) beginners, and (c) heavy smokers.



(a) (b) (c)

Figura 6: Sensitivity analysis of parameter β for population (a) potential, (b) beginners, and (c) heavy smokers.



(a) (b) (c)

Figura 7: Sensitivity analysis of parameter δ for population (a) potential, (b) beginners, and (c) heavy smokers.

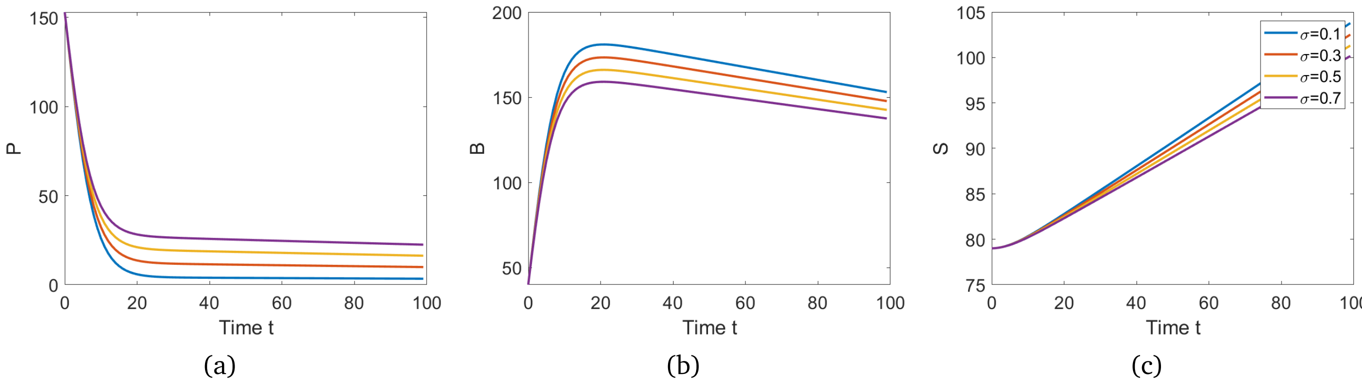


Figura 8: Sensitivity analysis of parameter σ for population (a) potential, (b) beginners, and (c) heavy smokers.

5. Conclusion

The discrete-time smoking model presented in this research is straightforward; nonetheless, it yields a complex analysis for examining the local stability of the positive equilibria. Nonetheless, we may still employ numerical simulations to investigate them. Sensitivity analysis of parameters associated with smoking behaviors provides crucial insights for managing the smoker population, emphasizing the need to reduce interactions among potential, beginner, and heavy smokers while maximizing self-control of smoking habits. This can be achieved through public health campaigns, regulatory measures, smoking cessation programs, and the reduction of marketing by tobacco businesses. The model presented in this paper may serve as a basis for future research by employing nonstandard difference schemes (as introduced by [24] and used in [25, 26]) or incorporating an additional control component into the model. Future research could also expand the model by taking into account outside variables like taxation, media influence, and anti-smoking campaigns; adding randomness to account for erratic social influences and policy changes could make the model more realistic; and expanding the model to multiple groups of smokers, influenced by social models like corruption [27], could shed light on how smoking spreads in various social and economic contexts.

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